MATH1510-I

Fundamental Theorem of Calculus

Fundamental Theorem of Calculus relates "solving $\frac{dF}{dx} = f$", or equivalently, "computing $\int f(x)dx$") to "Finding 'area under' the curve '$y = f(x)$'".

The result goes like this:

Theorem (Newton-Leibniz)

Assumption: $f: [a, b] \to \mathbb{R}$ is continuous

Conclusion:

1. The function $A(x)$ (also known as "area-finding function") given by $A(x) = \int_{t=a}^{t=x} f(t)dt$ is a solution of the differential equation $\frac{dF}{dx} = f$
2. Area under $y = f(x)$, where $a^* \leq x \leq b^*$ (a^*, b^* being any two numbers in (a, b)) is given by $F(b^*) - F(a^*)$, where $F(x) = \int f(x)dx$. (Of course, it is also equal to $A(b^*) - A(a^*)$)

Proof

We need some tools:

3. Integral mean value theorem. It says: the following:

   Assumption: $f: [a, b] \to \mathbb{R}$ is a continuous function.

   Conclusion: $\exists \xi \in [a, b]$ such that $\int_a^b f(x)dx = f(\xi) \cdot (b - a)$

   Remark: Another way of saying the same thing is:

   $\exists \xi \in [a, b]$ such that $\frac{1}{b-a} \cdot \int_a^b f(x)dx = f(\xi)$ which means "$f(\xi)$" is the average (or "mean") value of the integral.

(Step 1) Define the area-finding function $A(x) = \int_a^x f(t)dt$. First we show

$$A'(x) = f(x)$$

To do this, consider the difference-quotient, then take limit.

That is

$$\frac{A(x+h) - A(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

$$= \frac{f(\xi)h}{h}, \qquad \exists \xi \text{ between } x \text{ and } x+h$$

$$= f(\xi), \qquad \exists \xi \text{ between } x \text{ and } x+h$$

Next, by Sandwich/Squeeze Theorem, since ξ lies between x and $x+h$, as $h \to 0$, $\xi \to x$.

Therefore, we have $\lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)$

(Step 2) Idea is "compute the integration constant in two different ways"

Now we have two functions, (i) $A(x)$ --- the area-finding function, (ii) any function solving the D.E. $\frac{dF}{dx} = f(x)$.

In (Step 1), we showed that $A'(x) = f(x)$. This together with $F'(x) = f(x)$ leads to the conclusion that $F(x) - A(x) \equiv C$.

So now we compute C in two ways.
Evaluating at b^*, we get $F(b^*) - A(b^*) = C$
Evaluating at a^*, we get $F(a^*) - A(a^*) = C$

So we get

$$F(b^*) - \int_{\alpha}^{b^*} f(t)dt = C$$

$$F(a^*) - \int_{\alpha}^{a^*} f(t)dt = C$$

Now we choose $\alpha = a^*$ and get

$$F(a^*) - \int_{a^*}^{a^*} f(t)dt = F(a^*) - 0 \equiv C$$

Hence $C = F(a^*)$.

Putting this into

$$F(b^*) - \int_{a^*}^{b^*} f(t)dt \equiv C$$

we obtain

$$F(b^*) - \int_{a^*}^{b^*} f(t)dt = F(a^*)$$

Therefore we conclude that

$$F(b^*) - F(a^*) = \int_{a^*}^{b^*} f(t)dt$$

as requested.

Riemann Sum

Summary about Integrals

- Two types of integrals – indefinite integrals, definite integrals
- Indefinite integrals are written as $\int f(x)dx$. They are solutions of the Differential Equation $\frac{dF}{dx} = f(x)$. They are "unique" up to a constant.
- Indefinite integrals exists if f is a continuous function on $[a, b]$.
- Indefinite integral "improves" smoothness. Reason: if f is "continuous", then F is differentiable, so it's better.
- Definite integral are "area" (i.e. directed/signed area) "under" the curve $y = f(x)$.
- Fundamental Theorem of Calculus (FTC) says: "one can compute $\int_a^b f(x)dx$ using (i) find $\int f(x)dx$; (ii) give it a name, e.g. $F(x) \coloneqq \int f(x)dx$ ($\coloneqq$ means "left-hand side is the name of the right-hand side"); (iii) Compute the number $F(b) - F(a)$.
- It is not always possible to use FTC to find area. One important example is $\int_a^b e^{-x^2} dx$. This integral (actually "definite integral") comes up in Statistics. It cannot be computed using FTC, because there is a theorem saying that

> There is no "elementary" function F which satisfies the Differential Equation $\frac{dF}{dx} = e^{-x^2}$.

(An elementary function is, roughly speaking, any function one can build starting from * polynomials, trig. functions, exponential function, log function using $+, -, \times, \div, \sqrt[n]{\ }$ a finite number of times.)

- Because of the preceding bullet point, we sometimes still need to use something known as Riemann Sum.

Riemann Sum via an Example

The following simple example illustrates the idea of Riemann Sum.

Consider $f(x) = x, 0 \leq x \leq 1$.

Goal: Compute (via "approximation by rectangles") $\int_0^1 x dx$.

Solution:

Step 1) Partition $[0,1]$ into n subintervals. Many ways to do it. Simplest way – divide them into n equal-width subintervals, i.e.

$$x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \cdots, x_k = \frac{k}{n}, \cdots, x_n = 1$$

Step 2) Form the "approximate sum" (many ways to do it) by considering the right endpoint of each subinterval (*one can of course consider any other convenient point* ξ_k *in* $[x_{k-1}, x_k]$, *e.g. mid-point, left-endpoint, maximum point …*) , i.e.

$$\sum_{k=1}^{n} f(\text{right endpoint of } [x_{k-1}, x_k]) \cdot \underbrace{(x_k - x_{k-1})}_{\Delta x_k}$$

Step 3) The above line is actually $\sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} \frac{k}{n} \cdot \frac{1}{n} = \left(\frac{1}{n^2}\right) \sum_{k=1}^{n} k =$

$\left(\frac{1}{n^2}\right) \left(\frac{(n+1)n}{2}\right) = \left(\frac{n}{2n}\right) + \left(\frac{1}{2}\right)$

Step 4) Take limit $n \to \infty$ to get $\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = \frac{1}{2}$.

Remark:

If we reconsider the problem, we see that if we had $f(x) = x^p$, then the above computation could yield the answer $\int_0^1 f(x)dx = \frac{1}{p+1}$, provided we have the knowledge of the sum $\sum_{k=1}^{n} k^p$. However, such formulas are very complicated.

To overcome this, we can consider "non-equalwidth" partitioning.

Conclusions

4. Computing $\int_a^b f(x)dx$ using Riemann Sum is tedious.

5. Question: Why is it true that "if f is continuous on $[a,b]$, then $\frac{dF}{dx} = f(x)$ has solution"? Reason is this – (i) one can consider the "upper sum", which is

$$USum = \sum_{k=1}^{n} f(max.point\ in\ [x_{k-1}, x_k]) \cdot \Delta x_k$$

One can consider also the "lower sum", which is

$$LSum = \sum_{k=1}^{n} f(min.point\ in\ [x_{k-1}, x_k]) \cdot \Delta x_k$$

Then one can show that (under the Assumption that "f is continuous in $[a, b]$")

$$LSum \leq \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k \leq USum$$

for any convenient point ξ_k in $[x_{k-1}, x_k]$.

(ii) Now

$$USum - LSum = \sum_{k=1}^{n} \{f(max.point\ in\ [x_{k-1}, x_k]) - f(max.point\ in\ [x_{k-1}, x_k])\} \cdot \Delta x_k$$

Here $f(max.point\ in\ [x_{k-1}, x_k]) - f(max.point\ in\ [x_{k-1}, x_k])$ measures the "jump" (or "gap") between the "max value" and "min value" of f in the subinterval.

So if f is continuous, one can show that this number can be made as small as we wish (here one needs to use the concept of "uniform continuity").

Because of the above, one can show $|USum - LSum|$ can be made as small as one likes. As the Riemann sum is sandwiched between the $USum$ and the $LSum$, the above method shows that all 3 objects, i.e. the $USum$, the $LSum$ and the $Riemann\ Sum$ have the same limit, denoted by the symbol $\int_a^b f(x)dx$.

Functions Revisited

Recall that the most important object in "Calculus" is "function". In this course, we discussed simple functions such as polynomials, rational functions, trig. Functions, exponential functions, logarithm functions and their properties, such as their domains, ranges, whether they are increasing, decreasing etc.

Abstractly seen, a function is a "rule", i.e. $x \mapsto y$, where $y = f(x)$ (meaning "y is the value of the function computed/evaluated at x")

- Looking back, one should be able to notice that the trig. functions, log function, exp functions are defined by "infinite processes", whereas polynomial functions and rational functions are defined by "finite processes". The trig. functions and exponential functions, log functions are actually examples of "mathematical objects" known as power series.

E.g. $e^x = \sum_{k=0}^{\infty} \frac{(x-0)^k}{k!}$, $\ln(y) = (y-1) + \frac{(y-1)^2}{2} - \frac{(y-1)^3}{3} + \cdots$

(Here the number in "red" is called the center of the power series)

Questions:

1. The above power series look like Taylor series (i.e. the series one obtains when the error terms go to zero as $n \to \infty$). So, we can ask "do all power series come from power series of some well-known functions?")
2. Are there other infinite process methods to define functions?

Answers

1. The answer to Question 1 is "no". There are many power series, which are not related to any well-known functions;
2. As for Question 2, there are other ways to define "new" functions using "infinite processes". One way is by means of "integration".

An Example

The Gamma function given by $\Gamma(\alpha) \coloneqq \int_{x=0}^{x=\infty} x^{\alpha-1}e^{-x}dx$.

Issues

- This integral integrate from 0 to ∞, so the domain (of integration) is $[0, \infty)$.
- We would like to know how to differentiate such kind of functions (defined by integration)

  First we look at the "differentiation" issue.

Questions

As before, when given a function, we want to know how to compute its derivative.

- What is $\Gamma'(\alpha)$?
- Related question: What is $F'(v)$ if $F(v) \coloneqq \int_{u=p(v)}^{u=q(v)} f(v)dv$? (where the lower integration limit is a function of v. Also the upper limit is a function of v).

What is $F'(v)$ if $F(v) \coloneqq \int_{u=p(v)}^{u=q(v)} f(u,v)dv$? etc.