Limits

Introduction For us the reason why we talk about 'limits' is because when we defined derivatives, i.e. given a function $f : (a, b) \to \mathbb{R}$, and fix $c \in (a, b)$, and consider

$$\lim_{x\to 0} \frac{f(x) - f(c)}{x - c}$$

we have actually been looking at the 'limit' of the (new) function given by

$$g(x) = \frac{f(x) - f(c)}{x - c}$$

which is a function defined on the 'punctured interval' $(a, b)\backslash\{c\}$.

Some Assumptions

In the following, we 'tacitly agree' that when we write

$$\boxed{\text{Let } \boxdot \ \text{ be a real function}}\ ,$$

we mean (where the notation $\boxdot$stands for the 'name' of any function, such as $f, g, \tan, \ln$ etc.)

$$\boxed{\text{Let } \boxdot \ \text{ interval} \to \text{interval}}\ .$$

We will use the following definition.

Definition Let f be a real function and c, L be two real nos. The we say

$$\boxed{\text{the left-limit of } f \text{ at } c \text{ \underline{exists} and \underline{equals} } L}\ ,$$

and write it as $\boxed{\lim_{x\to c^-} f(x) = L}$ if the following two conditions hold:

1. f is defined on a 'punctured interval' of c, i.e. $(a, b)\backslash\{c\}$
2. If $\left\{\begin{array}{ccc} x & \to & c \\ \text{and} & & \\ x & < & 0 \end{array}\right.$, then $f(x) \to L$.

Remark Similarly, one knows that

$$\boxed{\lim_{x\to c^+} f(x) = L}$$

means

$$\boxed{\text{If } x \to c \ \&\ x > c, \text{ then } f(x) \to L}$$

Important Case $\boxed{\text{limit of } f \text{ at } c \text{ exists and equals } L}$ written

$$\boxed{\lim_{x\to c} f(x) = L}$$

if and only if both left-limit and right-limit exist and are equal to each other.

Remark For a limit to exist at c, the function doesn't have to be defined at c !
A good example of this is:

$$\lim_{x\to c} \frac{x^2 - c^2}{x - c}$$

Here, the function $\dfrac{x^2 - c^2}{x - c}$ is not defined at the point c !

Properties of Limits

The following list summarize some properties of limits, some of which we have already used (tacitly) in our computation of

$$\lim_{x\to c} \frac{x^2 - c^2}{x - c}$$

presented before.

1. (Uniqueness) Let f be a real function and c be a real no., then if the limit $\boxed{\lim_{x\to c} f(x)}$ exists, it must be unique. To say this more mathematically, we usually assume there were two limits, L_1 and L_2, and subsequently show that these two numbers are actually the same number. That is to say, we try to show

   $$\text{If } \lim_{x\to c} f(x) = L_1 \text{ and } \lim_{x\to c} f(x) = L_2, \text{ then } L_1 = L_2.$$

2. (i) (A constant function has same limit at different points.)
   Let $f : (a, b) \to \mathbb{R}$ be the function given by

   $$f(x) = L, \ \forall x \in (a, b)$$

   then $\lim_{x\to c} f(x) = L$. (or simply $\lim_{x\to 0} L = L$).

   (ii) (Identity function)
   Let $f : (a, b) \to \mathbb{R}$ be the function given by

   $$f(x) = x, \ \forall x \in (a, b)$$

   (such a function is called 'identity function'), then

   $$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$

3. $(+, -, \times, \div)$ All these are straightforward, and we list them below (assuming that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$):

(a) $\lim_{x\to c}(f \pm g)(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x) = L \pm M$,

(b) $\lim_{x\to c}(f \cdot g)(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = LM$,

(c) $\lim_{x\to c}(f/g)(x) = \lim_{x\to c} f(x)/\lim_{x\to c} g(x) = L/M$, provided $M \neq 0$

4. Let $\lim_{x\to c} f(x) = L$ and $L > 0$, then in a punctured interval of c, $f(x)$ has the same sign as L. More precisely,

$$\exists \epsilon > 0 \text{ s.t. } \forall x \in (c-\epsilon, c+\epsilon) \text{ we have } f(x) > 0.$$

- (This corrects a typo in the lecture !)

Remark Similar statement holds also when $L > 0$ is changed to $L < 0$.

5. Let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Suppose also that in a punctured interval of c, we have

$$f(x) \leq g(x),$$

then $L \leq M$.

6. ('Punctured Interval Theorem/Property') Let $f(x)$ and $g(x)$ be equal to each other on a punctured interval of c, then they have the same limit at c.

More precisely, Let $f(x) = g(x)$ on $(a, b)\backslash\{c\}$ & $\lim_{x\to c} f(x) = L$, then $g(x)$ satisfies

(i) $g(x)$ has limit at c,

(ii) the limit equals L (i.e. $\lim_{x\to c} g(x) = L$.

Remark Note that we haven't mentioned the limit of the 'composition of two functions', i.e. if $y = f(x)$ is a real function, $z = g(y)$ is another real function, then we can define another new function $k(x)$ by the rule $k(x) = g(y) = g(f(x))$ and consider its limit. (We will discuss this rule later!)

Example

Find $\lim_{x\to 1} \dfrac{x^2+2}{2x+1}$ using the rules of limits.

Solution To illustrate how the rules are used, let's write the solution steps out in detail.

Consider the ratio (or 'quotient') $\dfrac{x^2+1}{2x+1}$. Because at $x = 1$, the expression $2x+1$ is non-zero, we can use the rule for division of limits to get

$$\lim_{x\to 1} \frac{x^2+2}{2x+1} = \frac{\lim_{x\to 1}(x^2+2)}{\lim_{x\to 1}(2x+1)}$$

$$= \frac{\lim_{x\to 1} x^2 + \lim_{x\to 1} 2}{\lim_{x\to 1} 2x + \lim_{x\to 1} 1}$$

$$= \frac{(\lim_{x\to 1} x) \cdot (\lim_{x\to 1} x) + \lim_{x\to 1} 2}{(\lim_{x\to 1} 2)(\lim_{x\to 1} x) + 1}$$

$$= \frac{3}{3} = 1.$$

□

Two Simple but Useful Limits

Example 1 Consider the real function $f(x) = x^2$ defined on the open interval (a, b) and let $c \in (a, b)$. Compute (using limit definition) the derivative $\left.\frac{df}{dx}\right|_{x=c}$.

Solution Consider the quotient

$$\frac{x^2 - c^2}{x - c},$$

on any punctured open interval (a, b) containing the point c, then since $x - c \neq 0$, we can cancel the term $x - c$ to obtain

$$\frac{x^2 - c^2}{x - c} = x + c$$

Now apply the punctured interval theorem (i.e. the last property mentioned above!) to get

$$\lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} x + c = 2c.$$

Hence the $f'(c) = 2c$. $\square$

Example 2 Consider the real function $f(x) = x^n$ and any real no. c (Here n is assumed to be a natural no.). Compute (using limit definition) the derivative $\left.\frac{df}{dx}\right|_{x=c}$.

Solution Consider the quotient

$$\frac{x^n - c^n}{x - c},$$

on any punctured open interval $(a, b)\backslash\{c\}$. All we need to do is to study the limiting behavior of these quotients as $x \to c$.

Here one can use various methods to get the same answer. One way is to first introduce a new variable

$$h$$

by letting $x = c + h$ and consider the product of n copies of $(c + h)$, where $h \neq 0$. Doing this, we obtain

$$\frac{x^n - c^n}{x - c} = \frac{(c + h)^n - c^n}{h}. \tag{1}$$

Now multiplying everything out in the product $(c + h)^n$, we get the following

$$(c + h)^n = c^n + nh \times c^{n-1} + \text{a polynomial in } h \text{ starting from the term } h^2$$

Replacing $(c + h)^n$ by this expression in the numerator of the quotient

$$\frac{(c + h)^n - c^n}{h},$$

we obtain

$$\frac{(c + h)^n - c^n}{h} = \frac{nhc^{n-1} + [\text{polynomial in } h \text{ starting with the term } h^2]}{h}.$$

Cancelling the terms involving h and its higher powers we obtain

$$\frac{(c+h)^n - c^n}{h} = nc^{n-1} + [\text{polynomial in } h \text{ starting with the term } h]\,.$$

Now apply the punctured interval theorem to get

$$\lim_{h\to 0}\frac{(c+h)^n - c^n}{h} = \lim_{h\to 0}\left(nc^{n-1} + [\text{polynomial in } h \text{ starting with the term } h]\right)$$

$$= \lim_{h\to 0}\left(nc^{n-1}\right) + \lim_{h\to 0}(\text{polynomial in } h \text{ starting with the term } h)$$

$$= nc^{n-1}.$$

□

Arithmetic of Derivatives

Before we go on, we want to mention the following notations:

Notations

- $\Delta f = f(x) - f(c)$,
- $\Delta x = x - c$,
- $\dfrac{\Delta f}{\Delta x} = \dfrac{f(x)-f(c)}{x-c}$, (where $x \neq c$),
- $\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c} = \lim_{x\to c}\dfrac{\Delta f}{\Delta x} =$ (this is a no., not a quotient!) $\left.\dfrac{df}{dx}\right|_{x=c}$,
- We also denote $\left.\dfrac{df}{dx}\right|_{x=c}$ by $f'(c)$ (meaning 'the derivative of f calculated/evaluated at $x = c$.)

We say a function $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ if the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists (and is finite).

We have the following rules for $(+, -, \times, \div)$ of derivatives, plus the 'Chain Rule'.

In the following, we assume that f, g are both differentiable at c:

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$,
2. $(k \cdot f)'(c) = k \cdot f'(c)$, (where k is a constant),
3. $(f \cdot g)'(c) = f'(c)\cdot g(c) + f(c)\cdot g'(c)$, (product rule)
4. $\left(\dfrac{f}{g}\right)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$, provided $g(c) \neq 0$ (quotient rule)

If we assume that (i) f is differentiable at c, and (ii) g is differentiable at $f(c)$, then we have the following

Chain Rule

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Remark $g \circ f$ is the new function defined by the rule

$$\underbrace{(g \circ f)}_{\substack{\text{name}\\ \text{of the new}\\ \text{function}}}(x) = g(f(x)).$$

Remark Similarly, $f + g$ is the new function defined by the rule

$$\underbrace{(f + g)}_{\substack{\text{name}\\ \text{of the new}\\ \text{function}}}(x) = f(x) + g(x).$$

Remark The same idea works for $f - g$, $f \cdot g$ and f/g.

Proof of the Differentiation Rules

We proved only the Product Rule, because it involves a new idea, namely the idea that

'f is differentiable at $x = c$' implies 'f is continuous at $x = c$'.

We will describe more of this and the proof of some other rules in the next set of notes.

Summary

1. We introduced 'limit' to define 'derivative' of a function at a point $x = c$.
2. This limit is actually a function defined on a 'punctured interval'.
3. We introduced the arithmetic of limits.
4. We introduced the arithmetic of derivatives.
5. During the proof of the Product Rule, we found out that we need to use the property

   'f is differentiable at $x = c$' implies 'f is continuous at $x = c$'.