Keywords: Sup, Inf, Bounds, Limit

Notation: $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$

1. Let S be a non-empty subset of $\mathbb{R}$. Assume that $\sup(S)$ exists. Show then that it is 'unique'. (Hint: Suppose there were two of them, say s_1 and s_2. You should try to show that $s_1 = s_2$, hence unique!)

2. Let A be a non-empty subset of $\mathbb{R}$, b any positive real number. Show that $\sup(bA) = b \cdot \sup(A)$, where bA is the set defined by

$$bA = \{bx |\ x \in A\}.$$

3. Let a, b be two real numbers satisfying $a < b$. Show that $\sup(a, b) = b$ (the symbol on the left-hand side means "supremum of the set (a, b)".)

4. In each of the following questions, determine whether the set is "bounded" or "unbounded" (above and below!) and find (i.e. no need to prove!) an upper bound and a lower bound if you think it is bounded above (below). If you think it is unbounded above (below), show that this is so:

   (a) $\left\{ \frac{1}{x^5} \mid x \in \mathbb{R}\backslash\{0\}. \right\}$

   (b) $\left\{ \frac{1}{x|x|} \mid x \in \mathbb{R}\backslash\{0\} \right\}$

5. Consider the set $S = \left\{ \frac{2^n - 1}{2^n + 1} \mid n \in \mathbb{N} \right\}$. Is S bounded above (below)? Prove or disprove it by giving reasons.

6. Show, by using the $\epsilon - N$ definition of infinite limit, that

   (a) $\lim_{x\to\infty} \frac{x^{101} - 1}{x - 1} = \infty$

   (b) $\lim_{x\to\infty} e^x = \infty$
   (Hint: Show first that $e^x > 1 + x$, if $x > 0$ by using the definition of e^x).

   (c) Find, using your calculus knowledge, sup and inf of the following set:

$$S = \{y \mid y = (1 - x^2)^2, x \in [0, 2]\}$$