Assignment 3

Topics: Riemann Sum, Uniform Continuity, Uniform Convergence, Function Sequence, Series

Tools learned in other courses, which may be useful

- The 3 Mean Value Theorems
- L'Hôpital's Rule

Assignments

1. (Riemann sum) Let $0 < a < b$ and P be the partition of the interval $[a, b]$ given by the points

   $$x_0 = a, x_1 = a + \frac{(b-a)}{n}, x_2 = a + \frac{2(b-a)}{n}, \cdots, x_i = a + \frac{i(b-a)}{n}, \cdots, x_n = b.$$

   (a) Using this partition, write down the lower sum

   $$L(P, f)$$

   of the function $f(x) = x^2$.

   (b) Find the limit

   $$\lim_{\|p\| \to 0} L(P, f)$$

   Hint: The following formula may be useful: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

2. (Uniform Continuity) Show that the function

   $$f : \mathbb{R} \to \mathbb{R}$$

   given by

   $$f(x) = \arctan x$$

   is uniform continuous.
   Hint: Use one of the 3 value theorems, i.e. Rolle's Theorme, Lagrange's Mean Value Theorem or Cauchy's Mean Value Theorem.

3. (Uniform Continuity) Show that the function

   $$f : (0, 1) \to \mathbb{R}$$

   given by

   $$f(x) := \frac{1}{x^2}$$

   is not uniformly continuous.

4. (Uniform Continuity*) Show that the function

$$f : \mathbb{R} \to \mathbb{R}$$

given by

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ \\ 0 & x = 0 \end{cases}$$

is uniform continuous.

5. For each of the following function sequences, determine whether it is pointwise convergent or uniform convergent. Give reasons to explain.

   (a) $p_n(x) = nxe^{-nx^2}$

   (b) $q_n(x) = \dfrac{nx}{1 + n^2x^2}$, on $[\frac{1}{2}, 1]$

6. Determine whether the following series is pointwise convergent or uniformly convergent. Give reasons to explain.

$$\sum_{k=0}^{\infty} x^k(1-x)$$

where $x \in (-1, 1]$.