1. Apply Mathematical Induction:

①. Initial case $n=1$.

$\Rightarrow (f\cdot g)^{(1)} = (f\cdot g)' = f'\cdot g + f\cdot g' = \sum_{k=0}^{1} C_k^{1} f^{(k)} g^{(1-k)}$ just our product-rule,

the Leibniz's rule is true.

②. Assume that the $n=m$ is true, means we have:

$$(f\cdot g)^{(m)} = \sum_{k=0}^{m} C_k^m f^{(k)} g^{(m-k)} \text{ holds. } (*)$$

Now we discuss $n=m+1$ case:

$$(f\cdot g)^{(m+1)} = [(f\cdot g)^{(m)}]' \quad \text{(use our assumption (*) here)}$$

$$= [\sum_{k=0}^{m} C_k^m f^{(k)} g^{(m-k)}]'$$

$$= \sum_{k=0}^{m} C_k^m [f^{(k)} g^{(m-k)}]' \quad \text{(product-rule, } n=1 \text{ case)}$$

$$= \sum_{k=0}^{m} C_k^m [f^{(k+1)} g^{(m-k)} + f^{(k)} g^{(m+1-k)}]$$

$$= \underbrace{\sum_{k=0}^{m} C_k^m f^{(k+1)} g^{(m-k)}}_{t=k+1} + \sum_{k=0}^{m} C_k^m f^{(k)} g^{(m+1-k)}$$

$$= \sum_{k=1}^{m+1} C_{k-1}^m f^{(k)} g^{(m+1-k)} + \sum_{k=0}^{m} C_k^m f^{(k)} g^{(m+1-k)}$$

$$= \underset{\substack{\parallel \\ 1 = C_0^{m+1}}}{C_0^m} f^{(0)} g^{(m+1)} + \sum_{k=1}^{m} \underset{\substack{\parallel \\ C_k^{m+1}}}{\underbrace{(C_{k-1}^m + C_k^m)}} f^{(k)} g^{(m+1-k)} + \underset{\substack{\parallel \\ 1 = C_{m+1}^{m+1}}}{C_m^m} f^{(m+1)} g^{(0)}$$

$$= \sum_{k=0}^{m+1} C_k^{m+1} f^{(k)} g^{(m+1-k)}$$

Leibniz's rule still be true when $n=m+1$. so according to mathematical induction.

$$\sum_{k=0}^{n} C_k^n f^{(k)} g^{(n-k)} = (f\cdot g)^{(n)} \text{ holds for all } n \in N^+.$$

2. just expanding the bracket.

$a = 1\times 2 = 2.$ $\quad b = 1\times 3 + 2\times 2 = 7$. $\quad c = 1\times 1 + 1\times 2 + 2\times 3 = 9$

$d = 1\times 5 + 2\times 1 + 1\times 3 = 10$, $\quad e = 1\times 1 + 2\times 5 = 11$, $\quad f = 1\times 5 = 5.$

3. $(1+x+x^2+\cdots)(1-x+x^2-x^3+\cdots) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$LHS = \frac{1}{1-x}\cdot\frac{1}{1+x}$ (use the Taylor's formula from the reverse direction)

$= \frac{1}{1-x^2}.$

And RHS = A polynomial so we just need to expand $\frac{1}{1-x^2}$.

$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots = RHS = a_0 + a_1 x + a_2 x^2 + \cdots$ (use the geometric series)

so $a_n = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$

Remark : Some of you may ask the question about the "convergence radious" here, but actually what we do here just some "formal operation", don't relate to concrete number so we don't need to worry about convergence radius, you can google "generating function" for more.

Assignments.

1. $\int x^5 \sqrt{1+x^3}\, dx.$

Sol. (1) take $u = x^3 \Rightarrow du = 3x^2 dx.$

$\int x^5\sqrt{1+x^3}dx = \frac{1}{3}\int u\sqrt{1+u}\cdot du.$

(2) take $t = \sqrt{1+u} \Rightarrow dt = \frac{1}{2}\cdot\frac{1}{\sqrt{1+u}} du \Rightarrow 2t\,dt = du$, $u = t^2-1$

$\frac{1}{3}\int u\sqrt{1+u}\,du = \frac{2}{3}\int (t^2-1)\cdot t\cdot t\,dt = \frac{2}{3}\int (t^4-t^2)dt = \frac{2}{3}\cdot\frac{1}{5}\cdot t^5 - \frac{2}{3}\cdot\frac{1}{3}t^3 + \boxed{C}$

$= \frac{2}{15}(1+x^3)^{\frac{5}{2}} - \frac{2}{9}(1+x^3)^{\frac{3}{2}} + C$

2. Compute an approximation to $\int_0^1 \frac{\sin x}{x}\,dx$. with ereor less than $\underline{0.01}$

Sol: For the general case, in order to approximate $\int_0^1 f(x)\,dx$.

we use n-th order Taylor polynomial like:

$$\int_0^1 f(x)\,dx = \int_0^1 [T_n(x)+E_n(x)]\,dx = \underbrace{\int_0^1 T_n(x)\,dx}_{\text{our approximate value}} + \underbrace{\int_0^1 E_n(x)\,dx}_{\boxed{err}}.$$

$$\Rightarrow \left|\int_0^1 f(x)\,dx - \int_0^1 T_n(x)\,dx\right| = \left|\int_0^1 E_n(x)\,dx\right| \le \int_0^1 |E_n(x)|\,dx$$

we try to estimate it to give a bound to the $\underline{\text{error}}$.

Back to $\int_0^1 \frac{\sin x}{x}\,dx$.

~~En(x)~~

For $\sin(x) = x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \frac{\overset{(-1)^{n+1}\cos\xi}{(\sin x)^{(2n+3)}\big|_{x=\xi}}}{(2n+3)!}x^{2n+3}$

$$\Rightarrow \frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n} + \underbrace{\frac{(-1)^{n+1}\cos\xi}{(2n+3)!}x^{2n+2}}_{E_n(x).}$$

$$\Rightarrow \int_0^1 |E_n(x)|\,dx \le \frac{1}{(2n+3)!}\int_0^1 x^{2n+2}\,dx \quad (|\cos\xi| \le 1)$$

$$= \frac{1}{(2n+3)!}\cdot\frac{1}{2n+3} \le 0.01 = tol. \text{ is ok.}$$

when $n=0$. $\frac{1}{3!}\cdot\frac{1}{3} = \frac{1}{18} > 0.01$ not satisfied.

$n=1$. $\frac{1}{5!}\cdot\frac{1}{5} = \frac{1}{600} < 0.01$, satisfied.

so we just need to take $T_1(x) = 1 - \frac{1}{3!}x^2$ to approximate $\frac{\sin x}{x}$.

Actually. the exact value for $\int_0^1 \frac{\sin x}{x}\,dx = 0.946083\ldots$

and $\int_0^1 (1-\frac{1}{6}x^2)\,dx = 1 - \frac{1}{18} \approx 0.9444\ldots$

we can see the error appears in 3-rd digits.

3. $\int_0^{2\pi} x\,|\cos x|\,dx$

$$= \int_0^{\frac{\pi}{2}} x\cos x\,dx + \int_{\frac{3}{2}\pi}^{2\pi} x\cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} x\cos x\,dx$$

$$= (x\sin x + \cos x)\Big|_0^{\frac{\pi}{2}} + (x\sin x + \cos x)\Big|_{\frac{3}{2}\pi}^{2\pi} - (x\sin x + \cos x)\Big|_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \quad \left(\int x\cos x\,dx = x\sin x + \cos x + c\right)$$

$$= \frac{\pi}{2} - 1 + 1 - \left(-\frac{3}{2}\pi\right) - \left[-\frac{3}{2}\pi - \frac{\pi}{2}\right]$$

$$= 4\pi.$$

4. $I_n = \int_a^b x^k (\ln(x))^n\,dx$ (integrats · by parts)

$$= \frac{1}{k+1}\int_a^b (\ln(x))^n\,d\,x^{k+1}$$

$$= \frac{1}{k+1} x^{k+1}(\ln(x))^n\Big|_a^b - \frac{1}{k+1}\cdot\int_a^b x^{k+1}\cdot d(\ln x)^n$$

$$= \frac{x^{k+1}(\ln x)^n}{k+1}\Big|_a^b - \frac{1}{k+1}\cdot\int_a^b x^{k+1}\cdot n\cdot(\ln x)^{n-1}\cdot\frac{1}{x}\,dx.$$

$$= \frac{x^{k+1}(\ln x)^n}{k+1}\Big|_a^b - \frac{n}{k+1} I_{n-1} \qquad n \geq 1.$$