1. You can check this problem according to (4) (5) in last tutorial which said:

(4) $(u+v)' = u' + v'$

(5) $(ku)' = ku'$ k is a constant.

2. This problem shows a very important method to prove that a limit doesn't exist. Actually we have a theorem:

(Heine theorem) $\lim_{x \to x_0} f(x) = A \iff$ For all sequence $\{a_n\} \to x_0$, we have $\lim_{n \to +\infty} f(a_n) = A$.

this "$\iff$" told us if we can find 2 different sequences $\{a_n\}, \{b_n\} \to x_0$, but:

$\lim_{n \to +\infty} f(a_n) \neq \lim_{n \to +\infty} f(b_n)$, then means $\lim_{x \to x_0} f(x)$ doesn't exist. The idea of this argument comes from the uniqueness of limit.

Back to this one, we can choose:

$\{x_n = \frac{1}{2n\pi - \frac{\pi}{2}}\}, (n = 1, 2 \cdots)$, so $\{x_n\} \to 0$

$\{y_n = \frac{1}{2n\pi + \frac{\pi}{2}}\}, (n = 1, 2 \cdots)$, so $\{y_n\} \to 0$

But $\lim_{n \to +\infty} f(x_n) = \lim_{n \to +\infty} \sin(2n\pi - \frac{\pi}{2}) = -1$

$\neq$

$\lim_{n \to +\infty} f(y_n) = \lim_{n \to +\infty} \sin(2n\pi + \frac{\pi}{2}) = 1$

this shows $\lim_{x \to 0} \sin\frac{1}{x}$ doesn't exist then of course $f(x)$ is discontinuous at $x = 0$.

3. For checking continuity, first is the definition at this point, here $f(0) = b$.

then consider $\lim_{x \to 0} f(x)$, here:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos x}{x^2} \text{ (for } x \neq 0) = \lim_{x \to 0} \frac{2\sin^2\frac{x}{2}}{x^2} \overset{t = \frac{x}{2}}{=\!=} \lim_{t \to 0} \frac{1}{2} \left(\frac{\sin t}{t}\right)^2 = \frac{1}{2} \quad \left(\lim_{t \to 0} \frac{\sin t}{t} = 1\right)$$

here we consider $x \to 0^+, x \to 0^-$ at the same time for we know the important limit $\lim_{t \to 0} \frac{\sin t}{t} = 1$ is 2 sides, in other cases we may still have to consider 2 sides separately.

At last. the continuity implies:

$\lim_{x\to 0} f(x) = f(0) \Rightarrow \frac{1}{2}$ (LHS) $= b$ (RHS)

So $b = \frac{1}{2}$.

$f(x) = |x| \cdot x^2 = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$

so $f'(x) = \begin{cases} 3x^2 & x > 0 \\ -3x^2 & x < 0 \end{cases}$, for the intersection point $x=0$, we have to use definition to consider it.

$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{x^3}{x} = 0 = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{-x^3}{x}$.

so $f'(0)$ exists, equal to $0 \Rightarrow f'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0 \end{cases}$

similarly we can get $f''(x) = \begin{cases} 6x & x \geq 0 \\ -6x & x < 0. \end{cases}$

then for $f'''(x)$, we still have $f'''(x) = \begin{cases} 6 & x > 0 \\ -6 & x < 0 \end{cases}$

but now $x=0$ is a peak point, $\lim_{x\to 0^+} \frac{f''(x)-f''(0)}{x} \neq \lim_{x\to 0^-} \frac{f''(x)-f''(0)}{x}$ means $f'''(0)$ doesn't exist.

so $f'''(x)$ is discontinuous at $x=0$.

5. $f(x) = (1-x^2)^2 = 1-2x^2+x^4 \Rightarrow f'(x) = -4x+4x^3 = 4x(x+1)(x-1)$

so "strictly increasing", we need $f'(x) > 0 \Rightarrow x > 1$ or $-1 < x < 0 \Rightarrow (-1,0) \cup (1,+\infty)$

Appendix: Consider any $x \in (a,b]$, we can see:

(1) $f(x)$ is continuous in $[a,x]$;

(2) $f(x)$ is differentiable in (a,x)

so we apply Lagrange's MVT in $[a,x]$: $f(x)-f(a) = f'(\xi)(x-a) = 0 \Rightarrow f(x) = f(a)$ for all $x \in (a,b]$

($f'(x)=0$, for $\forall x \in (a,b)$)

$\xi \in (a,x) \subset (a,b)$

So $f(x) = f(a)$ $\forall x \in [a,b]$ is a constant function.