(1) $\left.\frac{de^x}{dx}\right|_{x=c} = \lim_{h\to 0} \frac{e^{c+h}-e^c}{h}$ (from definition)

$= \lim_{h\to 0} \frac{e^c\cdot e^h - e^c}{h}$ ($e^{c+h} = e^c\cdot e^h$ from property: $e^{u+v} = e^u\cdot e^v$)

$= e^c \lim_{h\to 0} \frac{e^h-1}{h}.$ (Actually $\lim_{h\to 0}\frac{e^h-1}{h}$ just is the derivative at $x=0$, so we can see for $f(x)=e^x$, we can use the information at $x=0$ to deduce all other points)

① $e^x = 1+x+\frac{x^2}{2!}+\cdots$ definition.

$\Rightarrow \frac{e^h-1}{h} = 1+\frac{h}{2!}+\frac{h^2}{3!}+\cdots$ all terms contains "h" except the first one.

so $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

② Apply "sand-wich" thm. we have following estimate:

$1+x \leq e^x = 1+x+\frac{x^2}{2!}+\cdots < 1+x+x^2+x^3+\cdots$

the RHS comes from $\frac{x^2}{2!} < x^2$ etc. for $2! > 1$, and:

$1+x+x^2+\cdots$ is a geometric series, we have formula like.

$1+x+\cdots+x^n = \frac{1-x^n}{1-x}$ for finite term $x\neq 1$.

$1+x+x^2+\cdots = \lim_{n\to+\infty}(1+\cdots+x^n) = \lim_{n\to+\infty}\frac{1-x^n}{1-x}. = \frac{1}{1-x}$ ($|x|<1$)

the above limit only has meaning when $|x|<1$. or this series would tend to infinity.

So we get: $1+x\leq e^x < \frac{1}{1-x} \Rightarrow 1\leq \frac{e^h-1}{h} < \frac{1}{1-h}.$

for $h\to 0$, so we can regard $|h|<1$, then above estimate is valid.

from "sand-wich" thm we know:

$\lim_{h\to 0}\frac{e^h-1}{h} = 1$

$\Rightarrow \left.\frac{de^x}{dx}\right|_{x=c} = e^c.$

(2) $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ (definition)

$= \lim_{h\to 0} \frac{f(h)-0}{h}$ (condition)

$= 1 = g(0)$

$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$

$= \lim_{h\to 0} \frac{g(h)-1}{h}$

$= 0 = -f(0)$

(3). $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{f(x)\cdot g(h) + f(h)\cdot g(x) - f(x)}{h}$ $(f(x+y) = f(x)\cdot g(y) + f(y)\cdot g(x))$

$= \lim_{h\to 0}\left[\frac{f(x)\,(g(h)-1)}{h} + g(x)\cdot\frac{f(h)}{h}\right]$

$= f(x) \underbrace{\lim_{h\to 0} \frac{g(h)-1}{h}}_{=0} + g(x)\cdot \underbrace{\lim_{h\to 0} \frac{f(h)}{h}}_{=1}$. initial condition.

$= g(x)$.

it's the same to apply $g(x+y) = g(x)\cdot g(y) - f(x)\cdot f(y)$ to get $g'(x) = -f(x)$

Remark: Actually here f, g just our $\sin(x)$ and $\cos(x)$, this is another abstract definition for these 2 functions.

$f(x+y) = f(x)\cdot g(y) + g(x)\cdot f(y) \Leftrightarrow \sin(x+y) = \sin x\cdot\cos y + \cos x\cdot \sin y$.

(4). $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{m(x+h)+\beta - mx - \beta}{h} = \lim_{h\to 0}\frac{mh}{h} = m$. all $x\in\mathbb{R}$.

(5) $(u+v)'(x) = \lim_{h\to 0} \frac{(u+v)(x+h) - (u+v)(x)}{h} = \lim_{h\to 0} \frac{u(x+h)-u(x)+v(x+h)-v(x)}{h} = u'(x) + v'(x)$

$(ku)'(x) = \lim_{h\to 0} \frac{(ku)(x+h) - (ku)(x)}{h} = k \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} = k u'(x)$.