Taylor's Theorem

Keywords:
Taylor's Theorem, Taylor polynomial, Taylor series, Remainder term, center.

Notation & Terminology:

- We denote by $TP_n(x,c)$ the Taylor Polynomial of a degree n centered at c.
- If $f:(a,b)\to\mathbb{R}$ is 'smooth' (see below for explanation of this vocabulary) and the error term E_n satisfies

$$\lim_{n\to\infty} E_n = 0$$

for each fixed x and center c ([1]), then

$$f(x) = \underbrace{f(c) + \frac{1}{1!}f'(c)(x-c)^1 + \frac{1}{2!}f''(c)(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n + \cdots}_{\text{Taylor series centered at } x=c}$$

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1. Write down $TP_n(x,c)$ for each of the following C^∞ ([2] functions (You are required to write down the general term (i.e. the term $a_n(x-c)^n$:

   (a) $1+x-3x^2$ at $c=1$

   (b) $\sqrt{x}$ at $c=1$

2. In each of the following questions, the function is a 'smooth' function and $\lim_{n\to\infty} E_n = 0$. Find the Taylor polynomial at the given 'center':

   (a) $f(x) = \frac{1}{1+2x}$ $c=0$

   (b) $g(x) = \frac{x}{1+2x}$ $c=0$ (Hint: Use (2a))

   (c) $h(x) = \frac{1}{(1+2x)^2}$ $c=0$ (Hint: Use (2a))

   (d) $k(z) = \frac{1}{(1-x)(2-x)}$ $c=0$

3. (Irrationality of e) This question will lead you to show that e is an irrational number on a step-by-step basis.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + E_n, \tag{1.1}$$
$$\text{where } 0 < E_n < \frac{3}{(n+1)!}.$$

   Suppose $e = \frac{p}{q}$. Let $n =$ the 'maximum' among the two numbers q and 3. Rearrange (1.1) to get

$$\left(\frac{p}{q} - (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!})\right) = E_n.$$

[1]Or more precisely, $E_n(x,c)$

[2]By C^∞ or 'smooth' function we mean a function which has ALL derivatives.

then

$$\left(\underbrace{n!\frac{p}{q}}_{I} - \underbrace{n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}_{II}\right) = n!E_n.$$

Answer the following questions:

(a) How do we get the error bound $\frac{3}{(n+1)!}$? Explain!

(b) Why are (I) and (II) both natural numbers?

(c) Show that

$$0 < n!E_n < \frac{3}{4}.$$

(d) Complete the proof of the 'irrationality of e'.