§59 Maximum Modulus Principle

Lemma (Gauss' mean value theorem)

If $f(z)$ is analytic inside & on $C_\rho = \{|z-z_0| = \rho\}$,

then
$$\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta .}$$

Pf: Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)\,dz}{z-z_0} \qquad C_\rho: z = z_0+\rho e^{i\theta},\ 0\le\theta\le 2\pi.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})\,\rho i e^{i\theta}\,d\theta}{\rho e^{i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta .$$

※

Lemma: Suppose that f is analytic in $\{|z-z_0|<\varepsilon\}$ and
$$|f(z)| \le |f(z_0)| \quad \forall\, z\in\{|z-z_0|<\varepsilon\}.$$
Then $f(z) \equiv f(z_0)$, $\forall\, z\in\{|z-z_0|<\varepsilon\}$.

Pf: $\forall\, 0<\rho<\varepsilon$, Gauss' mean value theorem

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta$$

$$\Rightarrow \quad |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,d\theta = |f(z_0)|$$

$$\Rightarrow \quad |f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\theta})|\,d\theta$$

$$\Rightarrow \quad \int_0^{2\pi}\left[|f(z_0)| - |f(z_0+\rho e^{i\theta})|\right]d\theta = 0$$

Since $|f(z_0)| - |f(z_0+\rho e^{i\theta})| \geq 0$ and cts,

$$|f(z_0)| - |f(z_0+\rho e^{i\theta})| \equiv 0 \quad \forall \theta \ \& \ \forall \rho$$

$$\therefore \quad |f(z)| = |f(z_0)| \quad \forall \ z \in \{|z-z_0|<\varepsilon\}.$$

Recall that an analytic function with constant modulus is constant (eg4 of §26) $\Rightarrow$ $f(z) = f(z_0)$. $\#$

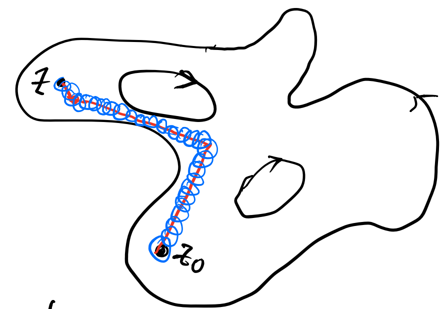
Thm (Maximum Modulus Principle)

If a function f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D, i.e. $\exists$ no point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ $\forall z \in D$.

Note: This is equivalent to: If f analytic in a domain D and $\exists z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ $\forall z \in D$. Then f is a constant function.

Pf: Suppose $\exists z_0 \in D$ s.t. $|f(z)| \leq |f(z_0)|$, $\forall z \in D$. Then $\forall z \in D$, connect z_0 to z by a polygon line L in D.

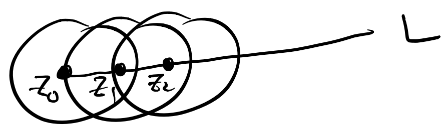
Let d = distance from L to ∂D. Then (by compactness of L)



there exists finitely many points $z_1, \cdots, z_{n-1}$ such that

$$B_d(z_0) \cup B_d(z_1) \cup \cdots \cup B_d(z_{n-1}) \supset L$$

and $z_1 \in B_d(z_0)$, $z_2 \in B_d(z_1), \cdots$, $z \in B_d(z_{n-1})$.



Then applying the lemma to $B_d(z_0) \Rightarrow f(z_1) = f(z_0)$

$\Rightarrow \quad |f(z_1)| \geq |f(s)|, \ \forall s \in B_d(z_1)$

$\Rightarrow f(z_2) = f(z_1) = f(z_0)$.

$\vdots$

and so on

we have $\quad f(z) = f(z_{n-1}) = \cdots = f(z_1) = f(z_0)$.

※

Cor: Suppose that a function f is cts on a closed and bounded region R and that it is analytic and not constant in the interior of R. Then the maximum value of $|f(z)|$ in R, which is always reach, occurs somewhere on the <u>boundary</u> of R and never in the interior.

(Pf: Immediately from maximum principle.)

<u>Note</u>: The corollary holds for <u>real part</u> and <u>imaginary part</u> of an non-constant analytic function as

f analytic, non-constant $\Rightarrow$ $g = e^{f}$ analytic, nonconstant

and $|g| = e^{\operatorname{Re} f}$.

(for Imaginary part $|e^{if}| = e^{\operatorname{Im} f}$)

Ch5 Series

§60 Convergence of Sequences

Def: (i) An infinite sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers has a limit z if $\forall \varepsilon > 0$, $\exists$ positive integer n_0 such that

$$|z_n - z| < \varepsilon, \quad \forall\, n > n_0.$$

(ii) When limit z exists, the sequence is said to converge to z and denoted by $\lim_{n\to\infty} z_n = z$.

(iii) If a sequence has no limit, it diverges.

Note: If limit exists, it is unique (Ex!)

Thm: Suppose that $z_n = x_n + i y_n$ $(n = 1, 2, 3, \cdots)$ & $z = x + iy$

Then $\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$.

Pf: ($\Rightarrow$) $\lim_{n\to\infty} z_n = z \Rightarrow \forall \varepsilon > 0, \exists\, n_0$ s.t. $|z_n - z| < \varepsilon$ $\forall n > n_0$

$$\Rightarrow \begin{cases} |x_n - x| = |\mathrm{Re}(z_n - z)| \le |z_n - z| < \varepsilon \\ |y_n - y| = |\mathrm{Im}(z_n - z)| \le |z_n - z| < \varepsilon \end{cases} \quad \forall\, n > n_0$$

$\therefore \lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$.

($\Leftarrow$) If $\lim_{n\to\infty} x_n = x$ & $\lim_{n\to\infty} y_n = y$, then

$\forall \varepsilon > 0$ (consider $\frac{\varepsilon}{2} > 0$ in the $\varepsilon\delta$ definition)

$\exists\, n_1 > 0$ s.t. $|x_n - x| < \frac{\varepsilon}{2}$, $\forall n > n_1$

$\exists\, n_2 > 0$ s.t. $|y_n - y| < \frac{\varepsilon}{2}$, $\forall n > n_2$

Then for $n_0 = \max\{n_1, n_2\}$, we have

$$|z_n - z| = |(x_n - x) + i(y_n - y)| \le |x_n - x| + |y_n - y|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad , \forall n > n_0$$

$\therefore \lim\limits_{n\to\infty} z_n = z$ ※

By the thm, one can write

$$\boxed{\lim_{n\to\infty}(x_n + i y_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n}$$

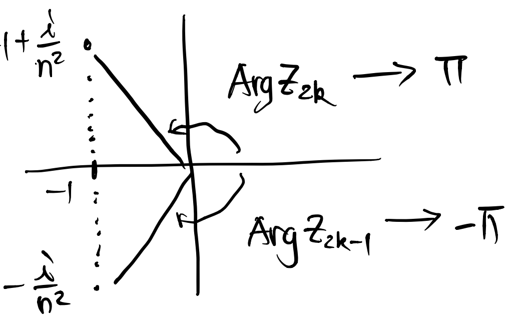
And hence $\boxed{\lim\limits_{n\to\infty}|z_n| = \left|\lim\limits_{n\to\infty} z_n\right|}$

eg: $\underbrace{\lim\limits_{n\to\infty}\left(-1 + i\frac{(-1)^n}{n^2}\right)}_{z_n} = \lim\limits_{n\to\infty}(-1) + i \lim\limits_{n\to\infty}\frac{(-1)^n}{n^2}$

$= -1$

Principal argument of z_n = $\mathrm{Arg}(z_n)$

if n even $z_n = -1 + \frac{i}{n^2}$



if n odd $z_n = -1 - \frac{i}{n^2}$

$\therefore \lim\limits_{n\to\infty} \mathrm{Arg}\left(-1 + i\frac{(-1)^n}{n^2}\right)$ doesn't exist !

In summary : If $z_n \to z$, then

$$\begin{cases} \operatorname{Re} z_n \to \operatorname{Re} z \\ \operatorname{Im} z_n \to \operatorname{Im} z \\ |z_n| \to |z| \end{cases}.$$

But $\qquad \operatorname{Arg} z_n$ may not converge!

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§61 <u>Convergence of Series</u>

<u>Def</u>: (i) An infinite <u>series</u> $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$

of complex numbers <u>converges</u> to the <u>sum</u> S if

the <u>sequence of partial sums</u>

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N, \quad N = 1, 2, 3, \cdots$$

<u>converges</u> to S, i.e. $\lim_{N \to \infty} S_N = S$.

We then write $\quad \sum_{n=1}^{\infty} z_n = S$.

(ii) When a series doesn't converge, we say that it <u>diverges</u>.

Clearly we have

<u>Thm</u> Suppose that $z_n = x_n + i y_n$ $(n = 1, 2, 3, \cdots)$ and

$$S = X + iY.$$

Then $$\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X \;\&\; \sum_{n=1}^{\infty} y_n = Y.$$

(Pf: Ex!)

Hence we can write

$$\boxed{\sum_{n=1}^{\infty}(x_n+iy_n)=\left(\sum_{n=1}^{\infty}x_n\right)+i\left(\sum_{n=1}^{\infty}y_n\right).}$$

<u>Cor1</u>: If a series of cpx numbers converges, then the n-th term converges to zero as n tends to infinity.
(Pf: Ex! using the above and the corresponding result in $\mathbb{R}$)

<u>Def</u>: A series $\sum_{n=1}^{\infty} z_n$ is said to be <u>absolutely convergent</u> if the (real) series $\sum_{n=1}^{\infty}|z_n|$ converges.

<u>Cor2</u>: The absolute convergence of a series of cpx numbers implies the convergence of that series.

<u>Pf</u>: If $\sum_{n=1}^{\infty} z_n$ converges absolutely, then $\sum_{n=1}^{\infty}|z_n|$ converges.

Since $0\le|x_n|\le|z_n|$ & $0\le|y_n|\le|z_n|$,

$\sum_{n=1}^{\infty}|x_n|$ & $\sum_{n=1}^{\infty}|y_n|$ converge by comparison test.

$\therefore$ $\sum_{n=1}^{\infty}x_n$ & $\sum_{n=1}^{\infty}y_n$ are absolutely convergent series in $\mathbb{R}$,

$\Rightarrow$ $\sum_{n=1}^{\infty}x_n$ & $\sum_{n=1}^{\infty}y_n$ converge in $\mathbb{R}$

$\Rightarrow$ $\sum_{n=1}^{\infty}z_n$ converges in $\mathbb{C}$. ##

<u>Terminology</u>: For a series $\sum_{n=1}^{\infty}z_n=S$ with partial sum

$S_N=\sum_{n=1}^{N}z_n$,

$\rho_n = S - S_N = \sum_{n=N+1}^{\infty} z_n$ is called the remainder after N terms of the series.

(For "series" starting from any finite integer $\sum_{n=l}^{\infty} z_n$ means the series $\sum_{k=1}^{\infty} z_{n-l+1}$.)

Then $S = S_N + \rho_N$ & $\rho_N \to 0$ as $N \to \infty$.

In fact, for any number S, one can let $\rho_N = S - S_N$, then

$$\sum_{n=1}^{\infty} z_n = S \iff |\rho_N| = |S - S_N| \to 0 \text{ as } N \to \infty.$$

eg: $\forall z$ s.t. $|z|<1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Pf: The Nth partial sum is

$$S_N(z) = 1 + z + \cdots + z^{N-1} = \frac{1-z^N}{1-z} \quad \left(\begin{array}{c}\text{since } |z|<1 \\ \Rightarrow z \neq 1\end{array}\right)$$

Let $S(z) = \frac{1}{1-z}$.

Then $\rho_N(z) = S(z) - S_N(z) = \frac{1}{1-z} - \frac{1-z^N}{1-z} = \frac{z^N}{1-z}$

$$\Rightarrow |\rho_N(z)| \le \frac{|z|^N}{1-|z|} \to 0 \text{ as } N \to \infty \quad (\text{since } |z|<1)$$

$\therefore \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. ※

§62 Taylor Series

Thm (Taylor's Theorem)

Suppose that a function f is analytic throughout a disk $|z-z_0|<R_0$ centered at z_0 with radius R_0. Then $f(z)$ has the power series representation

$$\boxed{f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad (|z-z_0|<R_0).}$$

We usually refer this as Taylor series expansion of $f(z)$ about the point z_0.

Notes: (i) f analytic at a point $z_0 \Rightarrow f$ analytic in a disk centered at z_0

$\Rightarrow f$ has a Taylor series expansion at z_0.

(ii) If f is entire, then f is analytic in $|z-z_0|<R_0$, $\forall R_0>0$.

$\Rightarrow$ the expansion becomes valid on $|z-z_0|<\infty$.

(iii) No convergence test is required as long as f is analytic in $|z-z_0|<R_0$.

(iv) For $z_0=0$, & f analytic in $|z|<R_0$, the series becomes

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n \quad (|z|<R_0)$$

which is the Maclaurin series.