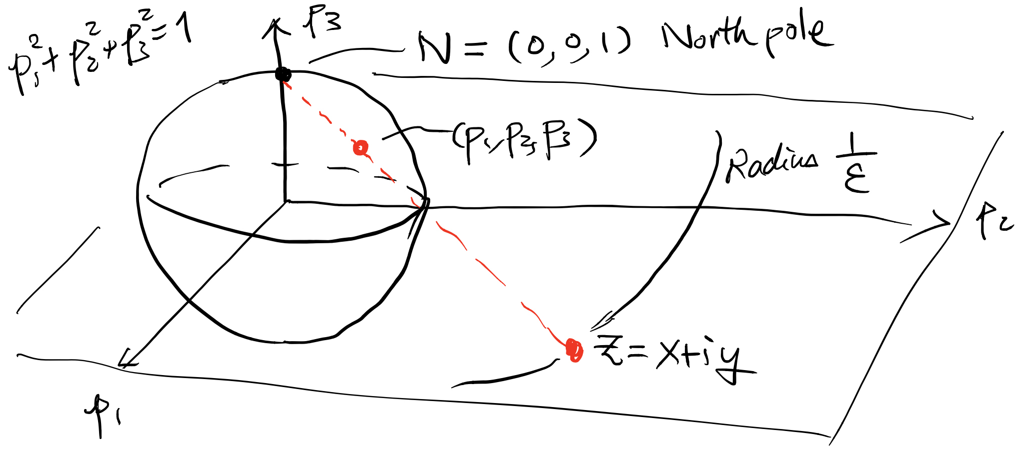
Stereographic projection:



Then $\begin{cases} z = x+iy = \dfrac{p_1+ip_2}{1-p_3} \quad \& \\ (p_1,p_2,p_3) = \left(\dfrac{2x}{|z|^2+1}, \dfrac{2y}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1} \right) \end{cases}$ (Ex!)

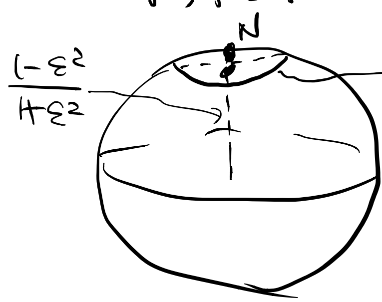
invertible, 1-1 & onto

$\therefore \quad S^2 \setminus \{N\} \longleftrightarrow \mathbb{C}$

Consider the (very large) circle $|z| = \frac{1}{\varepsilon}$ (for small $\varepsilon > 0$), we have

$z = \frac{1}{\varepsilon}(\cos\theta + i\sin\theta)$

$$(p_1,p_2,p_3) = \left(\frac{2\varepsilon\cos\theta}{1+\varepsilon^2}, \frac{2\varepsilon\sin\theta}{1+\varepsilon^2}, \frac{1-\varepsilon^2}{1+\varepsilon^2} \right) \to (0,0,1) \quad \text{(Ex)}$$

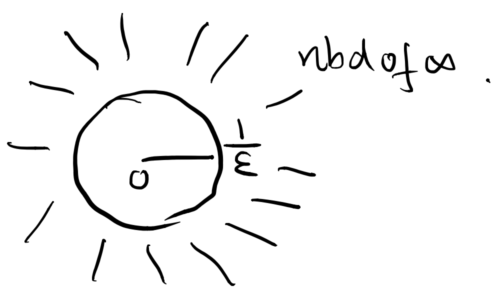


circle with radius $\frac{2\varepsilon}{1+\varepsilon^2} \longrightarrow 0$ as $\varepsilon \to 0$

$\therefore\ N=(0,0,1) \longleftrightarrow \infty$

Hence

Def: $\forall \varepsilon>0$, $\{|z|>\frac{1}{\varepsilon}\}$ is called a neighborhood of ∞, i.e. exterior of the closed disk of radius $\frac{1}{\varepsilon}$ is a nbd. of ∞.



Note that $\{|z|>\frac{1}{\varepsilon}\} = \{\frac{1}{|z|}<\varepsilon\}$, we have

Thm: If z_0 & w_0 are points in the z & w-planes respectively, then

(1) $\lim_{z\to z_0} f(z)=\infty$ if $\lim_{z\to z_0}\frac{1}{f(z)}=0$.

(2) $\lim_{z\to\infty} f(z)=w_0$ if $\lim_{z\to 0} f\left(\frac{1}{z}\right)=w_0$.

Moreover

(3) $\lim_{z\to\infty} f(z)=\infty$ if $\lim_{z\to 0}\frac{1}{f(\frac{1}{z})}=0$

Pf: (1) $\lim_{z\to z_0}\frac{1}{f(z)}=0$

$\Leftrightarrow \forall\ \varepsilon>0,\ \exists\delta>0$ s.t. $\left|\frac{1}{f(z)}-0\right|<\varepsilon,\ \forall\ 0<|z-z_0|<\delta$.

$\Leftrightarrow$ $\forall\, \varepsilon>0, \exists\, \delta>0$ s.t. $\underbrace{|f(z)|>\frac{1}{\varepsilon}}$, $\forall\, 0<|z-z_0|<\delta$.

$\Leftrightarrow$ $\lim\limits_{z\to z_0} f(z)=\infty$ $\qquad$ ε-nbd of ∞.

(2) & (3) Ex!

egs (1) Find $\lim\limits_{z\to -1} \dfrac{iz+3}{z+1}$

Soln: Consider $\lim\limits_{z\to -1} \dfrac{1}{\left(\frac{iz+3}{z+1}\right)} = \lim\limits_{z\to -1} \dfrac{z+1}{iz+3} \overset{\text{(Ex)}}{=} 0$

$\therefore \lim\limits_{z\to -1} \dfrac{iz+3}{z+1} = \infty$.

(2) Find $\lim\limits_{z\to\infty} \dfrac{2z+i}{z+1}$

Soln: Consider $(\zeta=\frac{1}{z})$ $\lim\limits_{\zeta\to 0} \dfrac{\frac{2}{\zeta}+i}{\frac{1}{\zeta}+1} = \lim\limits_{\zeta\to 0} \dfrac{2+i\zeta}{1+\zeta} = 2$

$\therefore \lim\limits_{z\to\infty} \dfrac{2z+i}{z+1} = 2$.

(3) Find $\lim\limits_{z\to\infty} \dfrac{2z^3-1}{z^2+1}$

Soln: Consider $(\zeta=\frac{1}{z})$ $\lim\limits_{\zeta\to 0} \dfrac{1}{\left(\frac{2(\frac{1}{\zeta})^3-1}{(\frac{1}{\zeta})^2+1}\right)} = \lim\limits_{\zeta\to 0} \dfrac{\zeta+\zeta^3}{2-\zeta^3} = 0$

$\therefore \lim\limits_{z\to\infty} \dfrac{2z^3-1}{z^2+1} = \infty$. ※

§18 Continuity

Def: A function f is <u>continuous</u> at a point z_0

if $\lim_{z \to z_0} f(z)$ exists, $f(z_0)$ exists

and $\lim_{z \to z_0} f(z) = f(z_0)$.

Note: Since "limits" are the same as in functions of z-variables, we have

Thm3 If $f(z) = u(x,y) + i v(x,y)$

Then $u(x,y), v(x,y)$ are continuous at (x_0, y_0)

$\Leftrightarrow$ f is continuous at $z_0 = x_0 + i y_0$.

(Pf omitted.)

Thm1: Composition of continuous functions is continuous.

Thm2: If f is continuous at z_0 & $f(z_0) \neq 0$,

then $f(z) \neq 0$ in some <u>nbd.</u> of z_0.

Thm4: If f is continuous on a region R that is both <u>closed and bounded</u>, then $\exists M > 0$

s.t. $|f(z)| \leq M, \forall z \in R$

where "equality" holds at least for one point.

§19 Derivatives

Def: Let f be a function whose domain of definition contains a neighborhood $|z-z_0|<\varepsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0},$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists.

Usual notations:
$$\begin{cases} \Delta z = z - z_0 \\ \Delta w = f(z_0+\Delta z) - f(z_0) \end{cases}$$

We often drop the subscript on z_0 & write

$$\Delta w = f(z+\Delta z) - f(z).$$

Then
$$\left(f'(z) = \right) \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

eg1: For $w = f(z) = \frac{1}{z}$, we have
$$\frac{dw}{dz} = -\frac{1}{z^2} \text{ or } f'(z) = -\frac{1}{z^2} \text{ (Pf: Ex!)}$$

eg2 : If $w = f(z) = \bar{z} = x - iy$

($u = x$, $v = -y$ are differentiable functions in x, y)

then $$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \frac{\overline{(\Delta z)}}{\Delta z}$$

$\therefore$ $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(\Delta z)}}{\Delta z}$ doesn't exist

(Hint: $= \begin{cases} 1 & \text{along } \Delta z = \Delta x + i0 \\ -1 & \text{along } \Delta z = 0 + i\Delta y \end{cases}$)

eg3 : $f(z) = |z|^2 = x^2 + y^2$ ($u = x^2 + y^2$, $v = 0$ are differentiable)

But $$\frac{\Delta w}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \bar{z} + \overline{\Delta z} + z \cdot \frac{\overline{(\Delta z)}}{\Delta z}$$

no limit

$$\Rightarrow \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \begin{cases} \text{doesn't exist}, & \text{for } z \neq 0 \\ 0, & \text{for } z = 0. \end{cases}$$

Notes: (1) eg3 $\Rightarrow$ function can be (cpx) differentiable at one point and non-differentiable elsewhere.

(2) Both $w = \bar{z} = x - iy$ & $w = |z|^2 = x^2 + y^2$ are differentiable mapping in real sense and

have partial derivatives of all order.

(3) As in real case, continuity $\not\Rightarrow$ differentiability

but differentiability $\Rightarrow$ continuity.

(Pf = Same as 1-real variable.)

§20 Rules for differentiation

If derivatives of f and g exist at z,

then

(1) $\frac{d}{dz} c = 0$, for const c.

(2) $\forall$ integer $n \geq 1$, $\frac{d}{dz} z^n = n z^{n-1}$

(3) $\frac{d}{dz}(f \pm g) = \frac{df}{dz} \pm \frac{dg}{dz}$

(4) $\frac{d}{dz}(fg) = f(z)\frac{dg}{dz} + \frac{df}{dz} g(z)$

(5) If $g(z) \neq 0$, then $\frac{d}{dz}\left(\frac{f}{g}\right) = \dfrac{g\frac{df}{dz} - f\frac{dg}{dz}}{g^2}$

Chain Rule: If f has derivative at z_0,

g has derivative at $f(z_0)$. Then $F(z) = g(f(z))$

has derivative at z_0 and $F'(z_0) = g'(f(z_0)) f'(z_0)$.

i.e. $\frac{dF}{dz} = \frac{dg}{dw}\frac{dw}{dz}$

§21 Cauchy-Riemann Equations

Thm: Suppose that $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the partial derivatives u_x, u_y, v_x, v_y exist at (x_0, y_0) and satisfy the

Cauchy-Riemann equation $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at (x_0, y_0)

Also, $f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$

Pf: Let $z_0 = x_0 + iy_0$ and $\Delta w = f(z_0 + \Delta z) - f(z_0)$

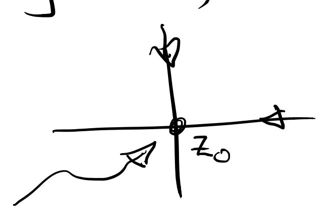
Write $\Delta z = \Delta x + i\Delta y$ ($\Delta x = \operatorname{Re} \Delta z$, $\Delta y = \operatorname{Im} \Delta z$)

Then

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\,\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}$$

By assumption $\lim_{z \to z_0} \frac{\Delta w}{\Delta z} = f'(z_0)$ exists,

$\therefore$ along any path of Δz going to 0, we have the same limit $f'(z_0)$.



In particular,

Horizontal approach $\quad \Delta z = \Delta x \quad (\Delta y = 0)$

$$\Rightarrow \quad f'(z_0) = \lim_{\Delta x \to 0} \left[\frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \, \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right]$$

$$= u_x(x_0, y_0) + i v_x(x_0, y_0)$$

(& u_x, v_x exist at x_0, y_0)

This already proved the last equality.

Vertical approach $\quad \Delta z = i\Delta y \quad (\Delta x = 0)$

$$\Rightarrow \quad f'(z_0) = \lim_{i\Delta y \to 0} \left[\frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i \cdot \frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{\Delta y} - i \, \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{\Delta y} \right]$$

$$= v_y(x_0, y_0) - i \, u_y(x_0, y_0)$$

Comparing the Real & Imaginary parts, we have

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$ which is the Cauchy-Riemann equation.

#

<u>Optional Ex:</u>

$f = u + iv$ cpx differentiable at z

$\Longleftrightarrow$ $F = \begin{pmatrix} u \\ v \end{pmatrix}$ real differentiable at (x,y)

with DF has the form $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

$\Longleftrightarrow$ $F = \begin{pmatrix} u \\ v \end{pmatrix}$ real differentiable at (x,y)

with DF has the form $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

with C-R eqt. $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\Longleftrightarrow$ u, v are real differentiable at (x,y)

and C-R eqt. $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

$\Longrightarrow$ $\exists$ partial derivatives u_x, u_y, v_x, v_y

$\not\Longleftarrow$ (next time) and C-R eqt $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$.