**MATH2550**
**Lecture 4.1**

**Notations:** (In the following, we will write a vector in the form $(u, v), (u, v)^t$ or $\begin{pmatrix} u \\ v \end{pmatrix}$ ). The notation $(u, v)^t$ is called "transpose" of the "row" vector $(u, v)$ and means simply that we rewrite the row vector "vertically", i.e. as $\begin{pmatrix} u \\ v \end{pmatrix}$ ).

**Keywords:** parametrization, tangent vectors, line integrals

**Main goals:**
To compute line integrals (2 kinds).

**Line Integral of Scalar Functions**
Line integral of the 1st kind (also known as "line integral of scalar function over a curve")

**Example.** Find $\int_C x\, ds$, where $C$ is a triangle formed by joining the points (0,0), (1,0) and (0,1) in that order.

**Method:**
1. Parametrize $C$. (This can be done in various ways. They should give the same numerical answer!) Why the same answer! Because the integral $\int_C xds$ depends ONLY on (a) the "geometry" of $C$ and (b) on the "integrand" $x$, i.e. it doesn't depends on how you parametrize (= "describe" using equations) $C$.
2. Next, we (i) restrict $f$ to be defined on $C$ (which means the variables $f$ depends on have to be on the curve $C$).
3. Then we compute $ds$ (here $ds$ means "infinitesimal (= infinitely short) length of the curve $C$, also known as "line element").
4. Then we compute the integral (from $a$ to $b$), where now the integration domain is now some interval [a,b].

**Solution:**
Regarding item 1 in the "Method" above, we can, for example, parametrize the right-angle triangle as follows:
- Let $\vec{\alpha}_1(t) = (t, 0)$, where the domain is now $[0,1]$.

- Let $\vec{\alpha}_2(t) = (1-t, t)$, where the domain is $[0,1]$.
- Let $\vec{\alpha}_3(t) = (0, t)$, where the domain is now $[0,1]$.

Regarding item 2 in the above, $x$ is now "restricted" (or you can say "required") to "move" on the curve $C$. This means the following (because $C$ consists of 3 curves):

- When we are thinking about the curve given by $\vec{\alpha}_1$, then $x = t$ since the $x$ −coordinate of the function is now $t$. (I wrote $\vec{\alpha}_1$ instead of $\vec{\alpha}_1(t)$, because I am thinking of the curve as one single object. It's not incorrect to write $\vec{\alpha}_1(t)$, if you want to emphasize what the variable of the function $\vec{\alpha}_1$is ).
- When we are on the curve $\vec{\alpha}_2$, the $x$ −coordinate is $1-t$, so again $x = t$.
- When we are on the curve $\vec{\alpha}_3$, the $x$ −coordinate is 0, so $x = 0$..

Item 3

We need to compute $ds$. Remember that $s$ is the "length/displacement" when traveling around the curves $\vec{\alpha}_1, \vec{\alpha}_2$ and $\vec{\alpha}_3$ (and $ds/dt$ is the "speed"). They are respectively given by

On the curve $\vec{\alpha}_1$, the speed is $ds/dt = \|\vec{\alpha}_1'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{d0}{dt}\right)^2} = 1$, giving $ds = 1dt = dt$

On the curve $\vec{\alpha}_2$, the speed is $ds/dt = \|\vec{\alpha}_2'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{\left(\frac{dt}{dt}\right)^2 + \left(\frac{d(1-t)}{dt}\right)^2} = \sqrt{2}$, giving $ds = \sqrt{2}\, dt$.

On the curve $\vec{\alpha}_3$, the speed is $ds/dt = \|\vec{\alpha}_3'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{\left(\frac{d0}{dt}\right)^2 + \left(\frac{dt}{dt}\right)^2} = 1$, giving $ds = 1dt = dt$.

Item 4 Now we can compute the integrals $\int_{\vec{\alpha}_1} xds$, $\int_{\vec{\alpha}_2} xds$ & $\int_{\vec{\alpha}_3} xds$.

They are respectively

$$\int_{\vec{\alpha}_1} xds = \int_0^1 t \cdot 1 \cdot dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}.$$

$$\int_{\vec{\alpha}_2} xds = \int_0^1 (1-t)\sqrt{2}\, dt = \sqrt{2}\left[t - \frac{t^2}{2}\right]_0^1 = \frac{\sqrt{2}}{2}.$$

And also, we have

$$\int_{\vec{\alpha}_3} xds = \int_0^1 0 \cdot 1 \cdot dt = 0.$$

Adding them together, we obtain

$$\int_C xds = \int_{\vec{\alpha}_1} xds + \int_{\vec{\alpha}_2} xds + \int_{\vec{\alpha}_3} xds = \left(\frac{1}{2}\right)(1+\sqrt{2}).$$

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**Line Integral of Vector Field**

This is also known as “line integral of the 2nd kind”, or “work done”.).

The basic Idea is to

1. approximate the curve $C$ by many line segments,
2. computing the work done along each line segment.
3. After doing these, we sum them up.
4. As the number of line segments increase to infinity, we get better and better approximations. The ultimate answer is given the notation

$$\int_C (P, Q) \cdot (dx, dy)$$

or (if we vectors in column form)

$$\int_C \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_C Pdx + Qdy$$

or (after parametrizing by $x = x(t), y = y(t)$)) in the form

$$\int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right) dt$$

.

Let us consider an example.

**Example:**

Let $\vec{F}(x, y) = (1/x, y)$ be a vector field in the 2-dimensional plane $\mathbb{R}^2$ and $C$ a circle of radius 2 centered at the origin. Find the work done if we travel along the circle once.

**Solution:**

Method:

1. Write down the integral, $W = \int_C \left(\frac{1}{x}, y\right) \cdot (dx, dy)$
2. The above expression is “incalculable”! So we have to parametrize the curve first,

then seek ways to compute it.

3. (There are many ways to parametrize the curve, all producing the same answer) For example, we can let $x = 2\cos t\ , 2\sin t\ , 0 \leq t \leq 2\pi$.

4. The integral becomes $\int_0^{2\pi}(\frac{1}{2\cos t}, 2\sin t)\cdot(d(2\cos t)\ , d(2\sin t\ ))$

$$= \int_0^{2\pi}(\frac{1}{2\cos t}, 2\sin t)\cdot(d(2\cos t)/dt\ , d(2\sin t\ )/dt)dt$$

$$= \int_0^{2\pi}(\frac{1}{2\cos t}, 2\sin t)\cdot(-2\sin t\ , 2\cos t)dt$$

$$= \int_0^{2\pi}(-\tan t + \cot t)\,dt = \infty$$

Lecture 4.2

**Keywords:** Double integrals
& how to compute them

Line integral of vector fields (or "work done" in physics language) is related to something known as "double integral" by a well-known theorem, called the Green's Theorem. Green's Theorem is the basis of two other important theorems, i.e. Divergence Theorem and Stokes' Theorem.

Before we can understand the "statement" of the Green's Theorem, we need to know what "double integral" is. We explain this concept in the following paragraphs.

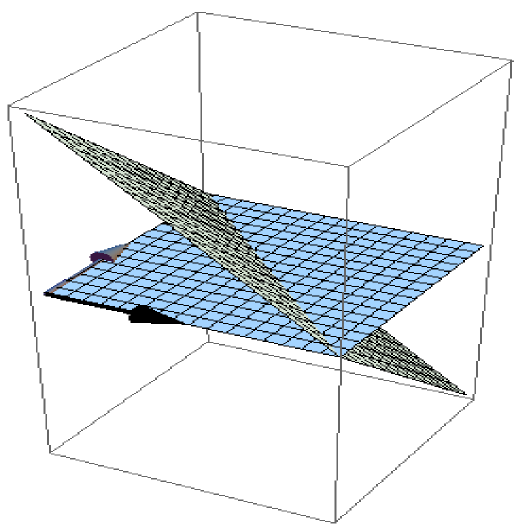
We start with a question.
**Question:** Given a function $f$ of two variables $x, y$, denoted by $f$ or by $f(x, y)$. How do we compute the volume "under" the surface $= f(x, y)$ ?

**Remark:** The word "under" doesn't have the usual sense of "under", as the following example shows.

**Example:**

$$f(x, y) = 1 - x - y$$

where the domain is $D = \{(x, y) |\ 0 \leq x \leq 1, 0 \leq y \leq 1\}$, then the surface looks like the following (black arrow = $x$-axis; grey arrow = $y$-axis):



The surface "under" the function $f(x, y) = 1 - x - y$ consists of two parts, (i) the part really above the triangular region including the origin and (ii) the part below the other triangular region farther away from the origin. (Intuitively, this two volumes cancel each other and will give a zero as answer).

**Suggested Solution:**

We compute the volume by considering the double integral of the form

$$\int_0^1 \int_0^1 f(x, y) dx dy$$

$$= \int_0^1 \int_0^1 (1 - x - y) dx\, dy$$

$$= \int_0^1 \left[x - \frac{x^2}{2} - xy + g(y)^1\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left[\left(1 - \frac{1^2}{2} - 1y + g(y)\right) - \left(0 - \frac{0^2}{2} - 0y + g(y)\right)\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(1 - \frac{1^2}{2} - 1y\right) dy$$

$$= \int_0^1 \left(\frac{1}{2} - y\right) dy = \left[\frac{1}{2}y - \frac{y^2}{2}\right]_0^1 = \frac{1}{2} - \frac{1}{2} = 0.$$

**Main Idea**

The main idea behind the computation of $\int_c^d \int_a^b f(x, y) dx dy$ is

(i) First slice the solid to get a slice at a certain "temporarily fixed value" of $y$.

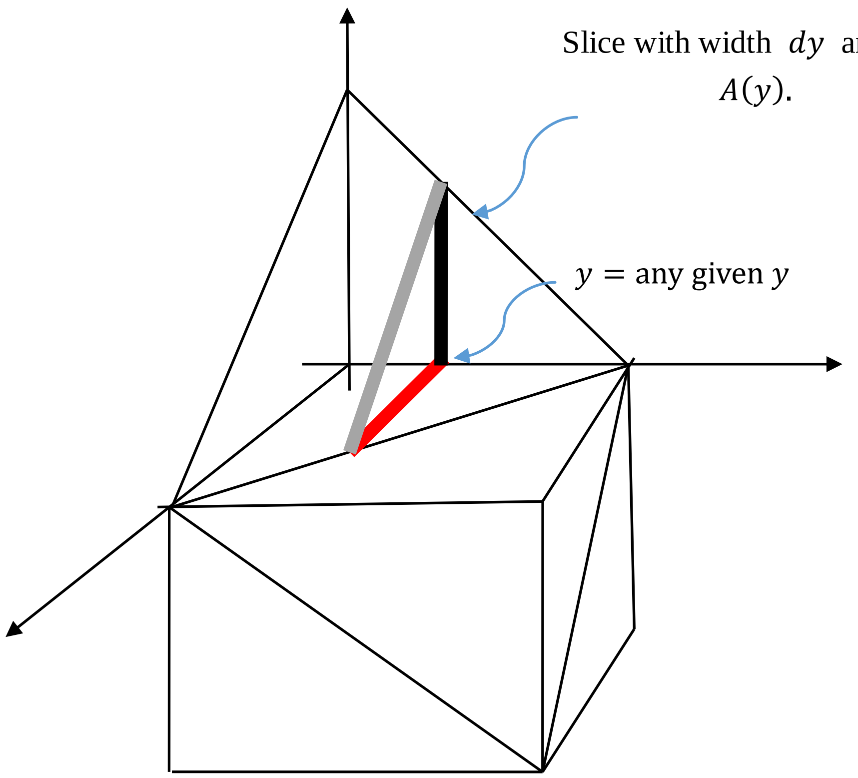
(ii) Give this slice a name $A(y)$.

(iii) Thicken this slice a little bit by multiplying by the (infinitely thin thickness) $dy$.

(iv) Step (iii) gives us the volume of the thickened slice, which is $A(y)dy$, integrating them with respect to $y$ gives then

$\int_c^d A(y)dy = \int_c^d \int_{x=a}^{x=b} f(x, y) dx\, dy$ (see the picture on the next page).

[1] We need to put a function which is "independent" of the variable $x$ here.

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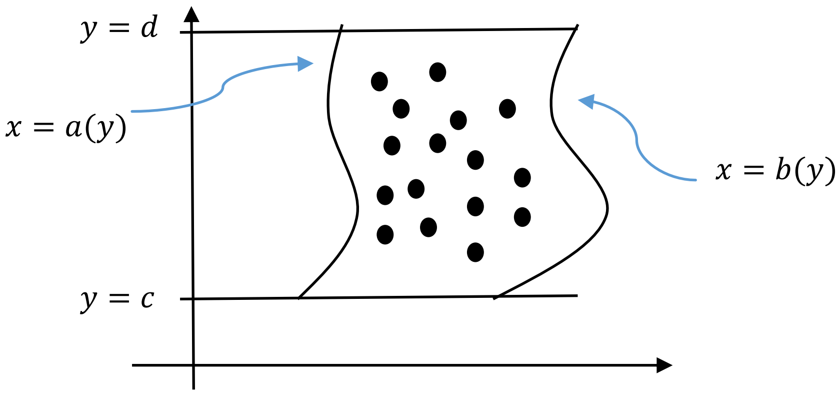
**Integration over More General Domains**

In the preceding discussion, we described how to compute the “double integral”, or “volume” under the surface $z = f(x, y)$, when $(x, y)$ belongs to a rectangular domain. Next we want to describe how to compute the “double integral” or “volume” under the surface $z = f(x, y)$, when $(x, y)$ belongs to more general domains.

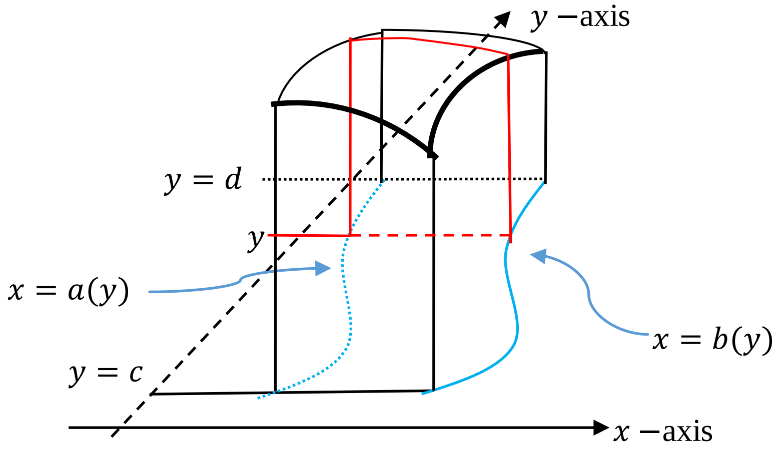
First, we make clear what kind of domains we have in mind.

**Definition:**

(Type I Domain). We call a domain $D$ in the plane a type I domain, if it looks like the following (i.e. the region with black circles):




To find the "volume" or double integral $\iint_D f(x, y)dxdy$, what we do is the same as before, except the following two points:



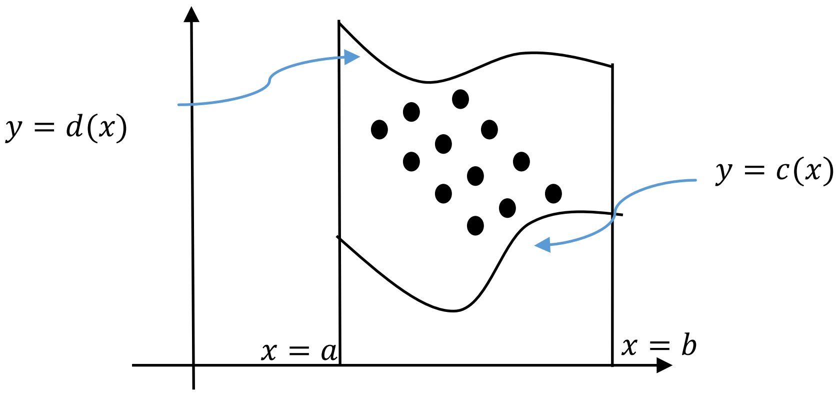

The only changes needed are:

(i) $A(y) = \int_{x=a(y)}^{x=b(y)} f(x, y)dx$ (i.e. the "area" of the red region!)

(ii) Volume $= \int_c^d A(y)dy = \int_c^d \int_{x=a(y)}^{x=b(y)} f(x, y)dx\, dy$

Instead of integrating over Type I domain, we can integrate over the following Type II domain.

(Type II Domain). We call a domain $D$ in the plane a type II domain, if it looks like the following (i.e. the region with black circles):




The only changes needed are:

(iii) $A(x) = \int_{y=c(x)}^{y=d(x)} f(x, y)dy$ (i.e. the "area" of the red region!)

(iv) Volume $= \int_a^b A(x)dx = \int_a^b \int_{y=c(x)}^{y=d(x)} f(x, y)dy\, dx$

## Lecture 5.1

**Green's Theorem**

Statement of the theorem:

Let $D$ be a domain in the plane, i.e. $\mathbb{R}^2$, let $\partial D$ denote the boundary curve of this domain (which can be shown to be a "closed" curve). Then the following formula holds:

$$\iint_D (Q_x - P_y)dxdy = \int_{\partial D} Pdx + Qdy$$

Here $P, Q$ are both functions of $x, y$ (i.e. functions of two variables).

**Remark:**

1. The right-hand side of this formula is incomputable. If one wants to compute it, one has to (i) parametrize $\partial D$, e.g. $\vec{\alpha}: [a, b] \to \mathbb{R}^2$ (where $\vec{\alpha}(t)$ represents a point on $\partial D$ at time $t$), (ii) then rewrites the right-hand side in the form

$$\int_a^b \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} dt$$

2. Another way to say this is to think of $\begin{pmatrix} P \\ Q \end{pmatrix}$ as a vector field, e.g. $\vec{F}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ as the velocity vector, i.e. $\vec{\alpha}(t)$ at time $t$. This way, the right-hand side takes the form

$$\int_a^b \vec{F} \cdot \vec{\alpha}'(t)dt$$

3. Sometimes we may want to velocity vector to be "normalized", i.e. having speed 1. To achieve this, we divide $\vec{\alpha}'(t)$ by its "speed", i.e. $\|\vec{\alpha}'(t)\|$ (denoted by the symbol $s$) and obtain

$$\int_{t=a}^{t=b} \vec{F} \cdot \vec{\alpha}'(t)dt = \int_{a^*}^{b^*} \vec{F} \cdot \left( \frac{\vec{\alpha}'(t)}{\|\vec{\alpha}'(t)\|} \right) \|\vec{\alpha}'(t)\|dt$$

(Note that the limits of integration have changed from the original $a, b$ to some "new" $a^*, b^*$)

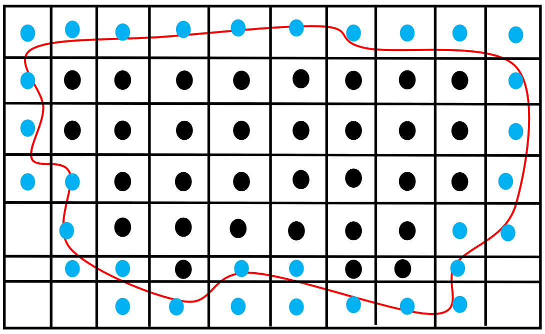
Usually you will see in textbooks that the highlighted part is equal to the line integral $\int_{s=a^*}^{s=b^*} \vec{F} \cdot \hat{\alpha}'(s)ds$, where $\hat{\alpha}'(s)$ is the unit tangent vector and $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ is the "line element" of the curve.

**Proof of G.T.**

IDEA: Use Fundamental Theorem of Calculus. One more idea is note that

> one only needs to prove Green's Theorem for the case when the domain is a rectangular box

This is because one can always put a domain $D$ inside a rectangle. After doing this, one can put as many grids as one wants inside the rectangle. Using such grids, one can "approximate" $D$ from "inside" and from "outside" using small rectangles. Look at the following diagram for an intuitive explanation.



**Notations:**

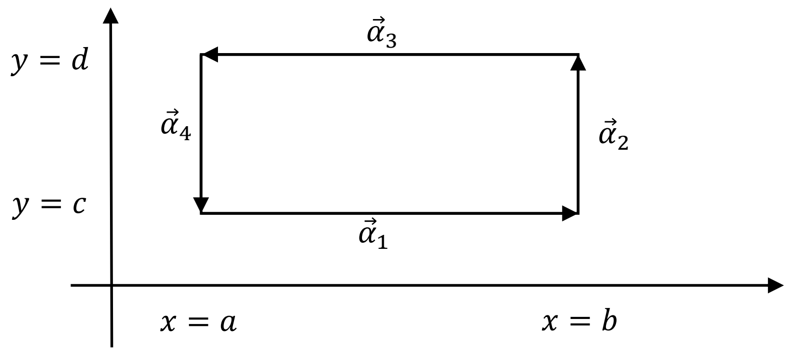
(Black dotted region) This is formed by rectangles inside the region bounded by the red curve.

(Black dotted/Blue dotted region) This is formed by rectangles inside (the region bounded by the red curve) and those rectangles where the red curve passes through.

**Remark:**

It can be shown that when the number of rectangles become larger and larger, both of these regions approximate the region bounded by the curve better and better.

From the above remark, we know that we "only" need to prove Green's Theorem on one rectangle. Consider the following rectangle:




**Goal:** Show that $\int_C Pdx + Qdy = \iint_D (Q_x - P_y)dxdy$

**Strategy:** We try to show that $\int_C Qdy = \iint_D Q_x dxdy$

Here $D$ is the rectangle joining the points $(a,c),(b,c),(b,d),(a,d)$ and $(a,c)$.
Method:

1 Consider the right-hand side. We have $\iint_D Q_x dxdy = \int_c^d \int_a^b Q_x dx\ dy$. By the usual Fundamental Theorem of Calculus, we have $\int_a^b Q_x dx = [Q(b,y) + fy-[Qa,y+fy]$, where $f(y)$ is some function depending "only" on the variable $y$.

2 Next, we try to show that $\int_c^d \int_a^b Q_x dx\ dy = \int_c^d \{[Q(b,y) + f(y)] - [Q(a,y) + fy]\}dy$ is equal to $CQdy$. To see this, first note that $\int_c^d \{[Q(b,y) + f(y)] - [Q(a,y) + f(y)]\}dy = \int_c^d [Q(b,y) - Q(a,y)]\, dy$.

3 Next, we compute $\int_C Qdy$. Here we have 4 curves – they are $\vec{\alpha}_1$, $\vec{\alpha}_2$, $\vec{\alpha}_3$ and $\vec{\alpha}_4$.

(i) $\vec{\alpha}_1(t) = (t,c)$, where $a \le t \le b$.

(ii) $\vec{\alpha}_2(t) = (b,t)$, where $c \le t \le d$

(iii) $\vec{\alpha}_3(t) = (a+b-t,d)$, where $a \le t \le b$ (because then when $t = a$, the $x-$ coordinate of $\vec{\alpha}_3(t)$ is equal to $b$. When $t = b$, the $x-$coordinated of $\vec{\alpha}_3(t)$ is equal to $a$)

(iv) $\vec{\alpha}_4(t) = (a, c+d-t)$, where $c \le t \le d$

4. Using (3), we get the following line integrals:

(i) $\int_{\vec{\alpha}_1} Qdy = 0$, $\int_{\vec{\alpha}_3} Qdy = 0$

(ii) $\int_{\vec{\alpha}_2} Qdy = \int_c^d Q(b,t)\cdot 1 \cdot dt = \int_c^d Q(b,y)\cdot 1 \cdot dy$ (after changing the "name" of the variable from $t$ to $y$)

(iii) $\int_{\vec{\alpha}_4} Qdy = \int_{t=c}^{t=d} Q(a,c+d-t)\cdot \frac{dy}{dt}\cdot dt = \int_c^d Q(b,c+d-t)\cdot(-1)\cdot dt$

$$= \int_{y=d}^{y=c} Q(b,y)\cdot(-1)\cdot(-dy) = \int_{y=d}^{y=c} Q(b,y)dy = -\int_{y=c}^{y=d} Q(b,y)dy.$$

(iv) Combining all four line integrals, we get what we wanted to prove.

**Green's Theorem using Normal Vector** (2D Divergence Theorem)

**Keyword:** normal vector

To understand the following discussions, you need to remember this:

(Green's Theorem says: "$\int_{\partial D} P^* dx + Q^* dy = \iint_D \left(Q_x^* - P_y^*\right) dxdy$," where $D$ denotes a domain in the plane, $\partial D$ denotes the "boundary curve" of this domain (this curve is a closed curve, i.e. it has no "beginning" and no "ending".)

Starting from this, we can prove the "2D divergence theorem", if we use "normal vector" instead of "tangent vector".

**Statement of the 2D Divergence Theorem:**

Let $\vec{F}(x,y) = \left(P(x,y), Q(x,y)\right)^t$ in the 2-dimensional plane $\mathbb{R}^2$, then

$$\iint_D \nabla \cdot \vec{F}\, dxdy = \int_{t=a}^{t=b} \vec{F} \cdot \vec{N}(x(t), y(t)) dt$$

where $(x(t), y(t))$ are the coordinates of a point on the boundary curve $\partial D$, $\vec{N}$ the outward pointing normal vector (of any length).

Proof:

We change the tangent vector $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$ to (outer) "normal vector", i.e. the vector $\vec{N}\left(x(t), y(t)\right) = \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix}$, then the line integral becomes

$\int_a^b \vec{F}\left(x(t), y(t)\right) \cdot \vec{N}\left(x(t), y(t)\right) dt =$

$$\int_a^b \vec{F}\left(x(t), y(t)\right) \cdot \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix} dt = \int_a^b \begin{pmatrix} P^*(x(t), y(t) \\ Q^*(x(t), y(t)) \end{pmatrix} \cdot \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix} dt$$

$$= \int_a^b [P\left(x(t), y(t)\right) y'(t) - Q\left(x(t), y(t)\right) x'(t)] dt$$

or in shorthand, $= \int_{\partial D} P dy - Q dx = \int_{\partial D} -Q dx + P dy.$

Applying the Green's Theorem to this expression, we obtain

$$\int_{\partial D} -Q dx + P dy = \iint_D (P_x - (-Q_y)) dxdy = \iint_D \nabla \cdot \vec{F} dxdy.$$

**Remark:**

We can make the highlighted part look nicer by "dividing by length of the normal vector", i.e.

$$\int_a^b \vec{F}\left(x(t), y(t)\right) \cdot \vec{N}\left(x(t), y(t)\right) dt =$$

$$\int_a^b \vec{F}\big(x(t),y(t)\big) \cdot \underbrace{\frac{\vec{N}\big(x(t),y(t)\big)}{\left\|\vec{N}\big(x(t),y(t)\big)\right\|}}_{\widehat{N}(x(t),y(t))} \underbrace{\left\|\vec{N}\big(x(t),y(t)\big)\right\| \, dt}_{ds}$$

In short form, one can write this as

$$\int_{\partial D} \vec{F} \cdot \widehat{N} \, ds$$

where now $\widehat{N}$ is the unit outward normal vector and $ds$ is the line element of the boundary curve $\partial D$.

**Remark:** This formulation will become useful, when we consider the 3D Divergence Theorem.