MATH 2550

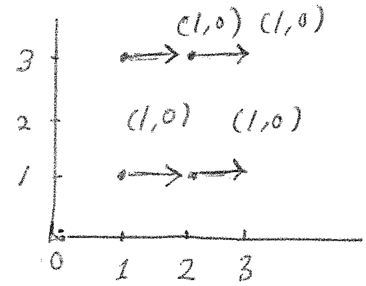
L2.

Vector Fields & Surfaces.

We introduced curves because they are related to flows. Another mathematical object related to flows is vector field (v.f. in short).

A planar vector field is a vector-valued function with 2 parameters/variables.

E.g. $\vec{v}(x,y) = (1,0)$ ⟵ This is vector field, which at each point (x,y) in $\mathbb{R}^2$ we have the same vector $(1,0)$ starting at (x,y).



More mathematically we can write

$$\begin{array}{ccc} \vec{v} : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x,y) & \longmapsto & (1,0) \\ & & \begin{array}{cc} \| & \| \\ p(x,y) & q(x,y) \end{array} \end{array}$$

$p(x,y)$ means the x-component of the vector $\vec{v} = (x,y)$ depends on the parameters/variables x & y.

Similar for $q(x,y)$.

p = x-component
q = y-component.

E.g. $\vec{v}(x,y) = (x,y)$

<u>Surfaces</u>

Surfaces are described mathematically in similar way to vector fields, i.e. they depend only 2 parameters.

The differences are:

① A surface lives in $\mathbb{R}^3$

② The vectors used are position vectors (for vector fields, the vectors are <u>NOT</u> position vectors!)

Mathematically, a surface in $\mathbb{R}^3$ is described by

domain in $\mathbb{R}^2$, will be explained later!

$$\begin{array}{ccc} \vec{\alpha} : D & \longrightarrow & \mathbb{R}^3 \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (u,v) & \longmapsto & \underbrace{(x(u,v),\ y(u,v),\ z(u,v))}_{\parallel} \end{array}$$

$\vec{\alpha}(u,v)$ if you prefer vector notations.

E.g. The plane $z = 1 - x - y$.

One can also use the more "symmetric" way of writing, i.e.

$x + y + z = 1$.

<u>RMK</u>: Just like $y = mx + c$ (in school math) represents a line in $\mathbb{R}^2$.

power 1

constants.

$$z = Ax + By + C$$

constants

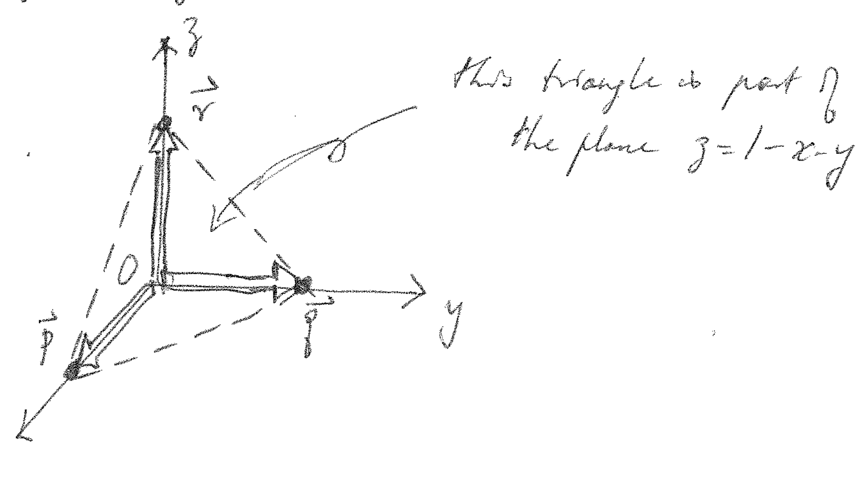
power 1

represents a plane in $\mathbb{R}^3$.

Since $z = 1 - x - y$ represents a plane, three non-collinear points determine a plane, we can sketch $z = 1 - x - y$ provided that we can find 3 such points.

One way to find them is to choose simply
$\vec{p} = (1, 0\,0)$, $\vec{q} = (0, 1, 0)$ & $\vec{r} = (0, 0, 1)$ which are three points on the plane $z = 1 - x - y$.

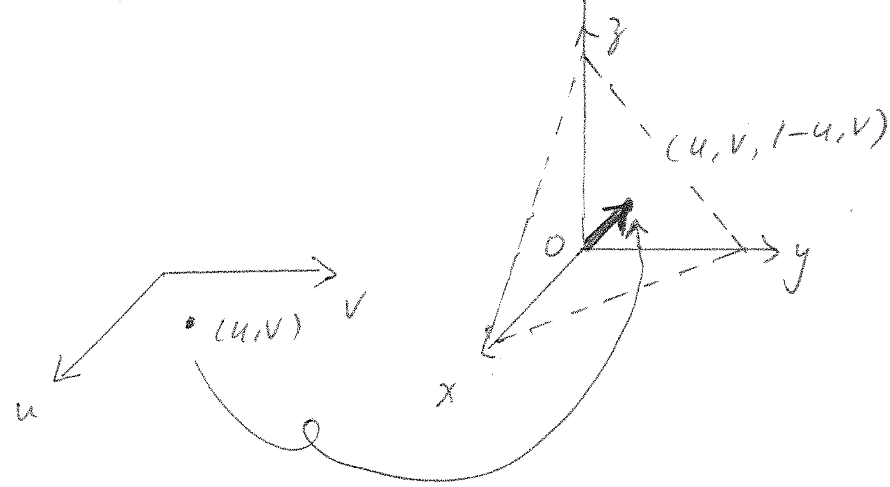
Then we have the following picture.



When we think about it, we see that either we understand it as
(i) To each (x, y) in the xy-plane, one can assign a height z $(= 1 - x - y)$.

or

(ii)
$$\begin{array}{ccc} \vec{\alpha} : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ \cup & & \cup \\ (u, v) & \longmapsto & (u, v, 1 - u - v) \end{array}$$



Definition of Surface

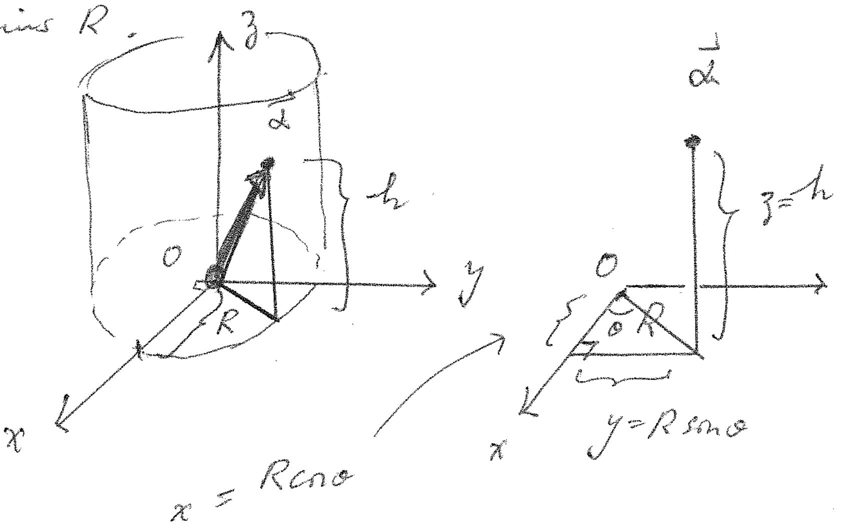
A surface is a vector-valued function $\vec{\alpha}$ from a domain D in $\mathbb{R}^2$ to $\mathbb{R}^3$, i.e.

$$\begin{array}{ccc} \vec{\alpha}: D & \longrightarrow & \mathbb{R}^3 \\ \Downarrow & & \Downarrow \\ (u,v) & \longmapsto & \vec{\alpha}(u,v) \end{array}$$

RMK: The domain D will usually be clear from the context, e.g. $D =$ a rectangle, the entire xy-plane, a disk etc.

E.g. Cylinder of radius R.

The position vector of the point $\vec{\alpha}$ on the cylinder depends on 2 parameters, θ & h.



More precisely,

$$\vec{\alpha} = \vec{\alpha}(\theta, h) = (R\cos\theta,\ R\sin\theta,\ h)$$

the 2 parameters.

RMK: $\vec{\alpha} = \vec{\alpha}(\theta, h)$ means "the position vector $\vec{\alpha}$" is actually "a function of θ & h".