Hyperbolic Solid Geometry

The Half-space Model

Def: Let $\mathbb{U} = \{ q = t + xi + yj : t, x, y \in \mathbb{R}, y > 0 \}$

be the upper half-space.

Let $\mathbb{M}$ be the full Möbius group

$$Tq = (aq+b)(cq+d)^{-1}$$

where a, b, c, d are complex numbers s.t.

$$ad - bc = 1$$

(complex $u + vi$, complex $i \leftrightarrow$ quaternion i)

The pair $(\mathbb{U}, \mathbb{M})$ models <u>3-dim'l hyperbolic geometry</u>.

Note: One needs to show that for $q \in \mathbb{U}$,

then $Tq \in \mathbb{U}$

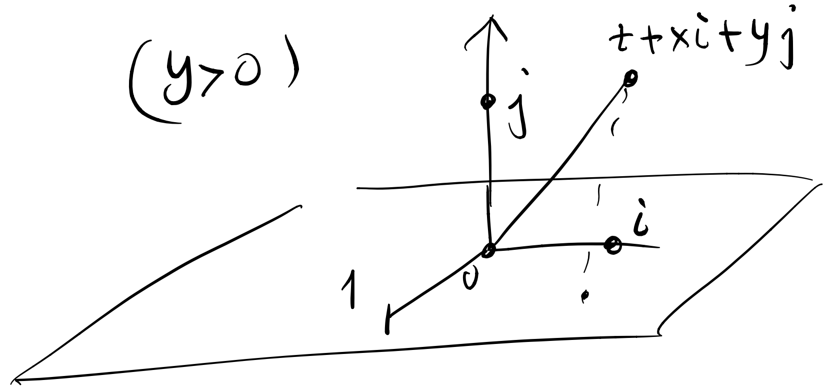
(Pf: Omitted, in fact, if $q = z + yj$, $z \in \mathbb{C}$, $y > 0$

then $Tq = (|q|^2 a\bar{c} + b\bar{d} + b\bar{z}\bar{c} + azd) + yj \in \mathbb{U}$)

Comparision:

	hyperbolic plane geometry	hyperbolic solid geometry
points	$x+yi$, $y>0$ upper half plane	$(t+xi)+yj$, $y>0$ $= z+yj$ $(z=t+xi\in\mathbb{C})$ upper half-space
group	Möbius transformation $\leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$ with $a,b,c,d \in \underline{\underline{\mathbb{R}}}$	Möbius transformation $\leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad-bc=1$ with $a,b,c,d \in \underline{\underline{\mathbb{C}}}$

$(y>0)$



Ideals Elements:

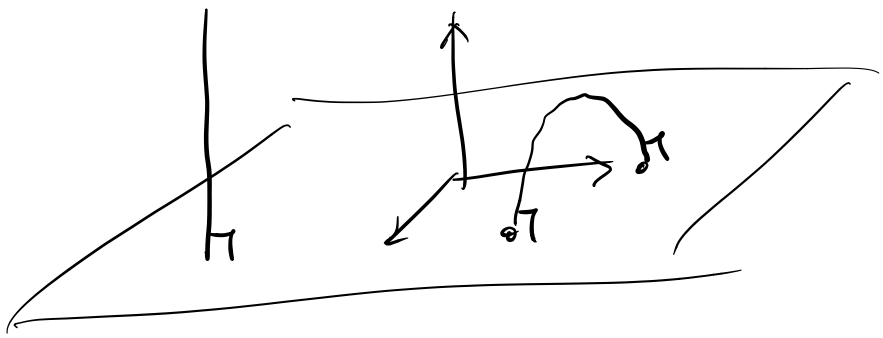
$z=t+xi \in \mathbb{C}$ ideal points (points at infinity)

$(\& \infty \in \hat{\mathbb{C}})$

Planes and Lines

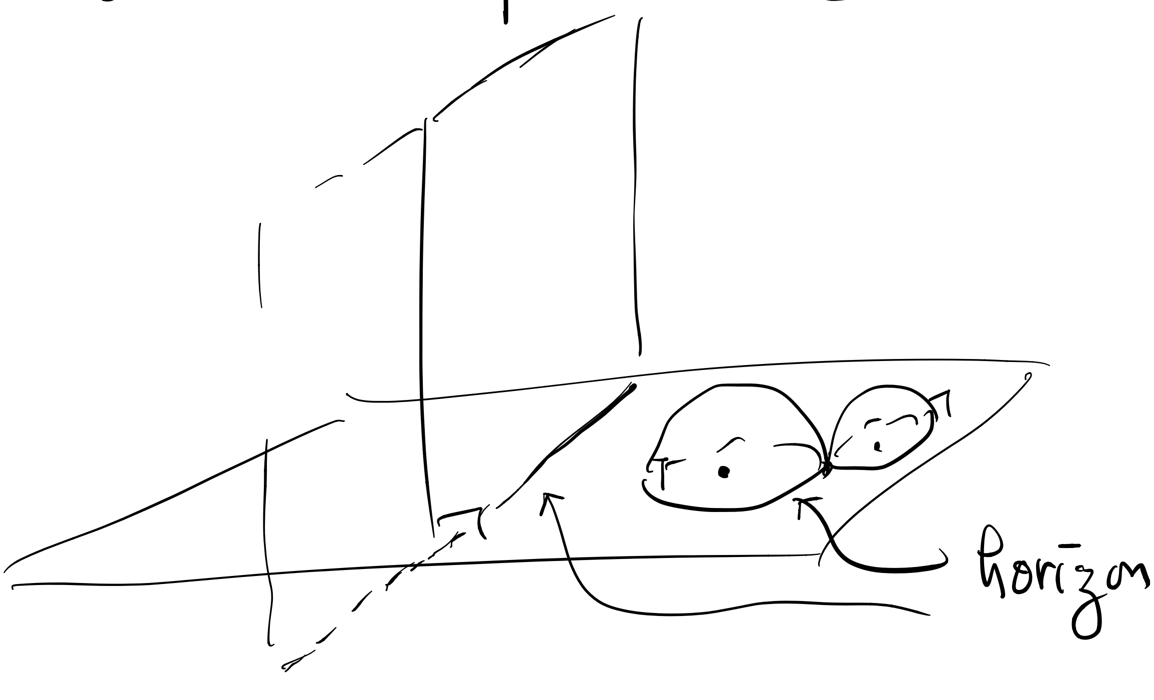
Hyperbolic straight lines

= half circle or Euclidean straight line in $\mathbb{U}$
perpendicular to the "plane at infinity" (tx-plane)



Hyperbolic plane

= Euclidean hemisphere a half-plane
perpendicular to the plane at infinity



The intersection of a hyperbolic plane with the plane at infinity is called the <u>horizon</u> of the plane.

<u>Parallelism</u>

- hyperbolic planes intersect
  $\Rightarrow$ intersection = hyperbolic line
- hyperbolic planes <u>do not</u> intersect
  (i) <u>parallel</u>: horizons are tangent
  (ii) <u>hyperparallel</u>: otherwise

<u>Cycles and Spheres</u>

<u>Cycle</u> = Euclidean circle a straight line in $\mathbb{U}$ that is <u>not</u> perpendicular to the plane at infinity

(hyperbolic circles, horocycles, and hypercycles as in 2-dim.)

Similarly, <u>sphere</u>, <u>horosphere</u> & <u>hyperspheres</u>.

= Euclidean spheres and planes that are not perpendicular to the plane at infinity.

<u>Arc-length</u>: γ : $\gamma(s) = t(s) + x(s)i + y(s)j$

(s = parameter, $a \le s \le b$)

$$\boxed{L(\gamma) = \int_a^b \frac{\sqrt{(t'(s))^2 + (x'(s))^2 + (y'(s))^2}}{y(s)}\, dt}$$

$$\boxed{\text{Volume of a solid } R = \iiint_R \frac{dt\,dx\,dy}{y^3}}$$

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Brief introduction to Elliptic Geometry

Elliptic geometry "=" spherical geometry.

(i.e. the natural geometry of the sphere)

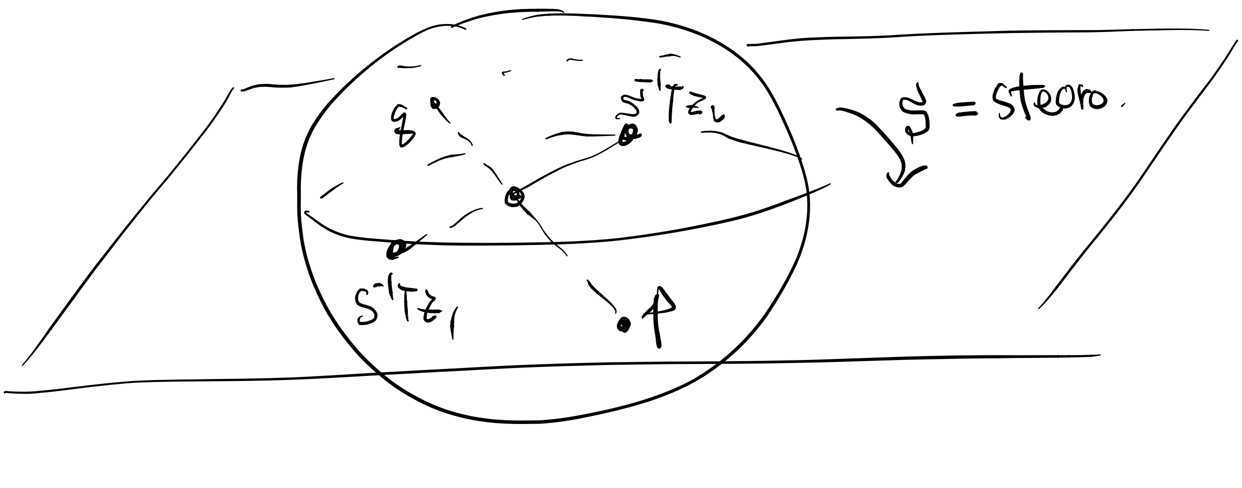
Def: The set

$$\mathcal{S} = \left\{ T \in \text{Möb} : Tz = e^{i\theta} \frac{z - z_0}{1 + \bar{z}_0 z}, \ \text{for some } \theta \in \mathbb{R}, \ z_0 \in \hat{\mathbb{C}} \right\}$$

is called the *elliptic group*.

The pair $(\hat{\mathbb{C}}, \mathcal{S})$ models "elliptic geometry".

The defining properties of T is



p, q are end points of a diameter,

then $z_1 = \mathcal{S}p$, $z_2 = \mathcal{S}q$ are complex number.

$\Rightarrow$ Tz_1, Tz_2 are also complex.

sphere

$\mathcal{S}^{-1}Tz_1$ & $\mathcal{S}^{-1}Tz_2$ are points on S^2

Then "if $T \in \mathcal{S}$, then $\mathcal{S}^{-1}Tz_1$ & $\mathcal{S}^{-1}Tz_2$ are also end points of same diameter.

i.e. p, q end points of a diameter

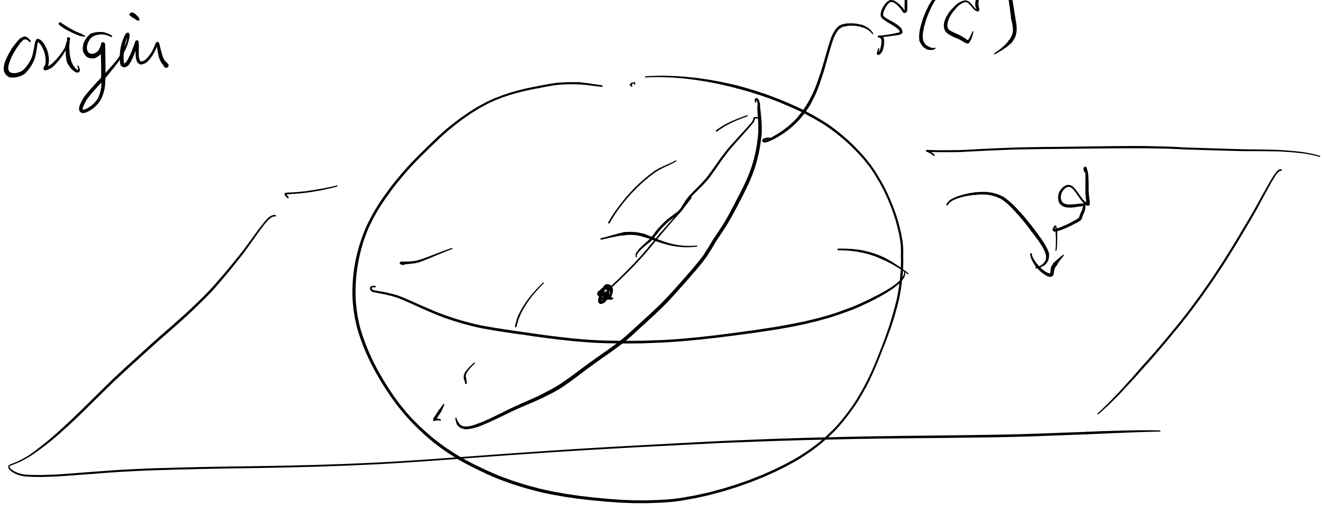
$\Rightarrow$ $\mathcal{S}^{-1}T\mathcal{S}p$ & $\mathcal{S}^{-1}T\mathcal{S}q$ are ends points of same diameter.

<u>Def</u>: In the model $(\hat{\mathbb{C}}, \mathcal{S})$ of elliptic geometry, a <u>great circle</u> is a circle C in the complex plane such that if $z \in C$, then $-\frac{1}{\bar{z}} \in C$.

(i.e. $\mathcal{S}^{-1}z \in C \Rightarrow$ diametrical opposite point

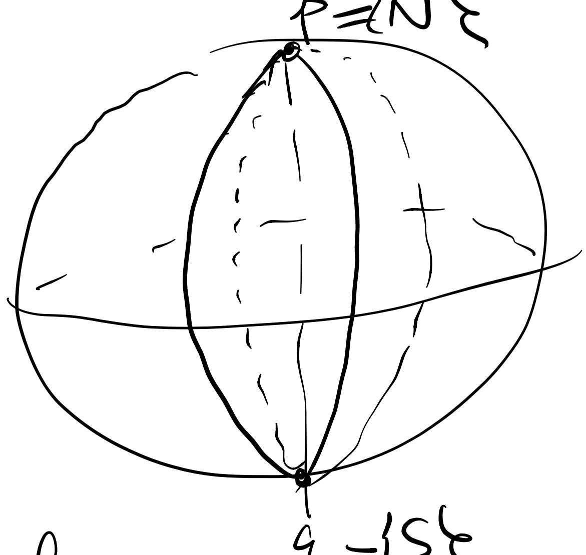
$S^{-1}(-\frac{1}{z}) \in C$)

$\Rightarrow$ $S^{-1}(C)$ is the intersection of the unit sphere with a plane passing thro the origin



An <u>elliptic straight line</u> is an arc of great circle.

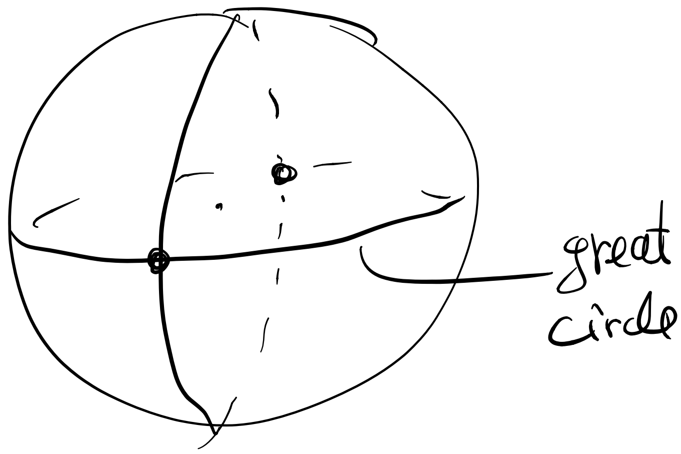
Then infinitely many great circles passing through the North pole and South pole.



So postulate 1 (of Euclidean geometry) fails in elliptic geometry.

Good news is "there is no "parallel" lines" in elliptic geometry since any two great circles intersect.

Postulate 5 fails too.



To make it a non-Euclidean geometry, we need to do "quotient".

"Single" Elliptic Geometry:

identify 2 diametrically opposite points at one single "point" in an abstract space.

Mathematically: if $z, z^d \leftrightarrow$ diametrical opposite points on the sphere.

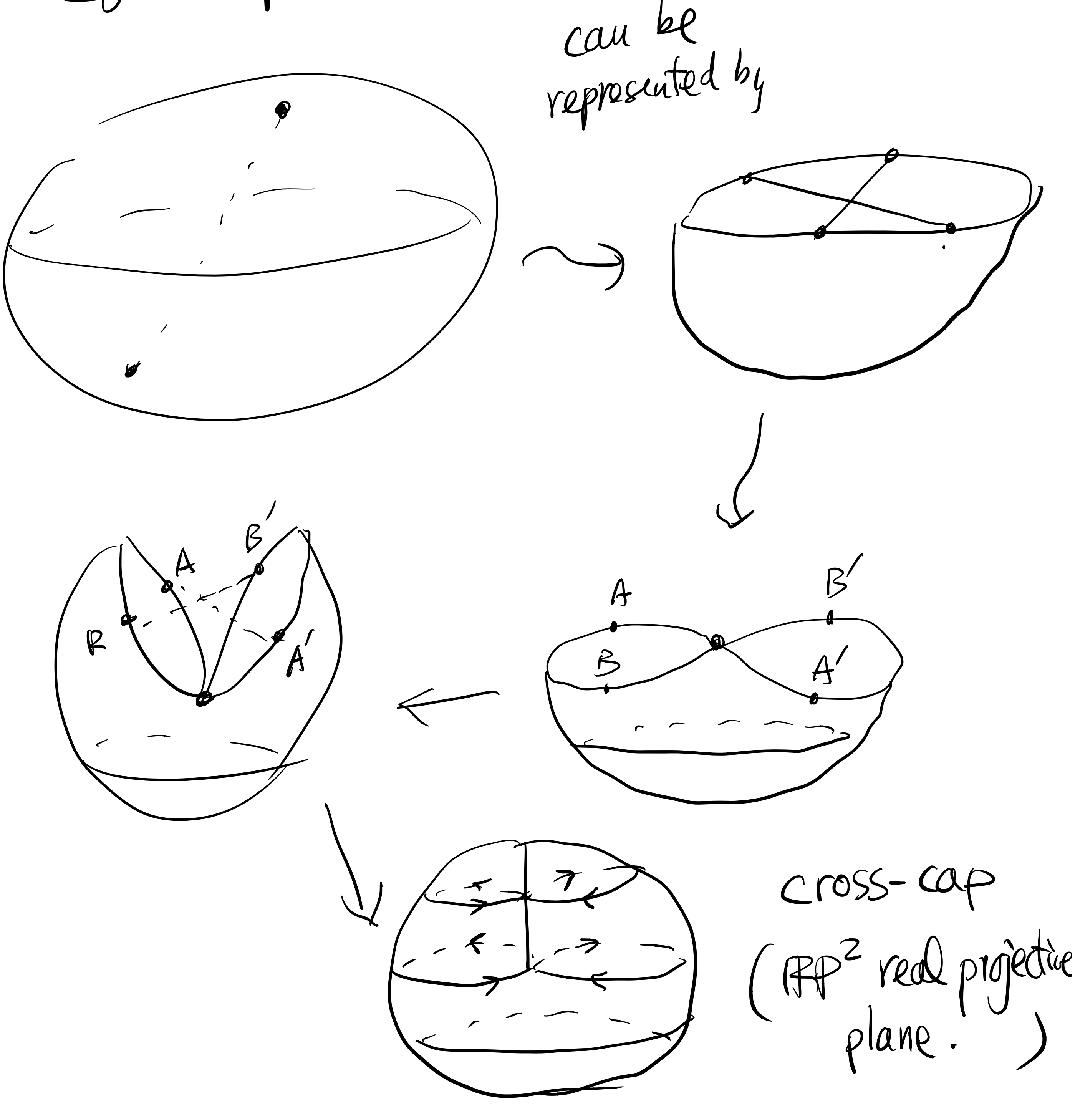
then $$\mathbb{C}/d = \{ [z] = \{z, z^d\} : z \in \mathbb{C} \}$$

= set of diameters of S^2.

One can check that

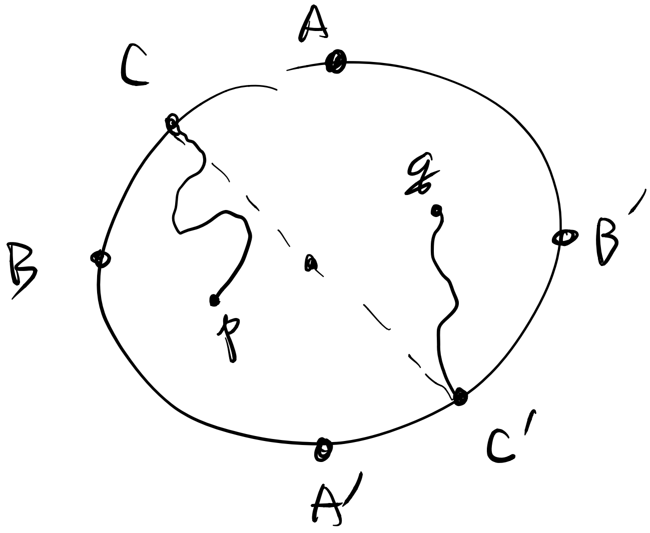
$(S/_d, \mathcal{S})$ forms a geometry (called

<u>single elliptic geometry</u>).



Another model:

Disk model



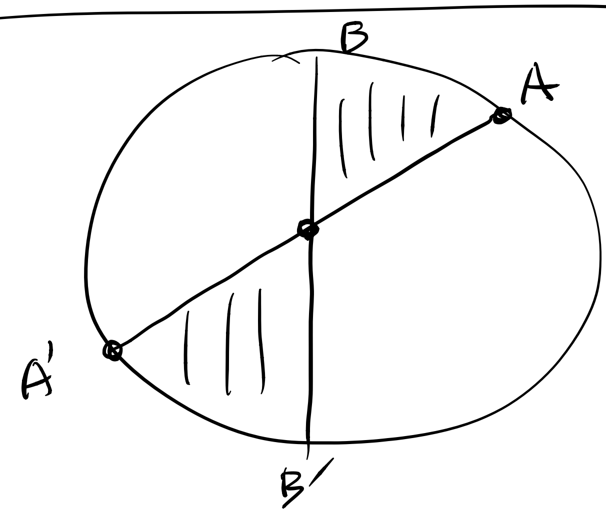
Distance and Area in Elliptic Geometry

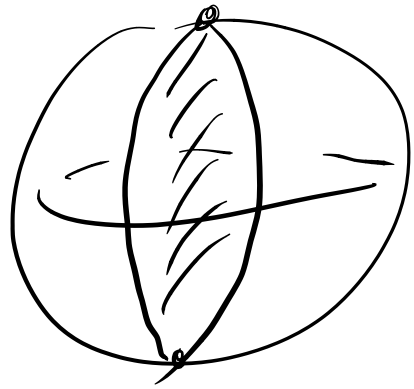
Arc-length $l(\gamma) = \int_a^b \frac{2|z'(t)|}{1+|z(t)|^2}\,dt$

$$\text{Area}(R) = \iint_R \frac{4r\,dr\,d\theta}{(1+r^2)^2} \quad \left(\text{in polar coordinates}\right)$$

$$= \iint_R \frac{4\,dx\,dy}{(1+x^2+y^2)^2}$$

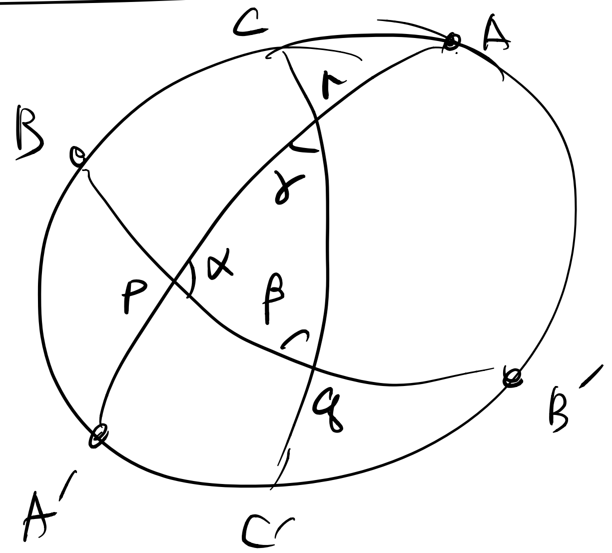
2-gons in elliptic geometry





Triangles :

We have



$$\boxed{\text{Area of } \Delta pqr = \alpha+\beta+\gamma-\pi}$$

angular excess

$$\boxed{\text{Sum of interior angles of a triangle in elliptic geometry} > \pi}$$