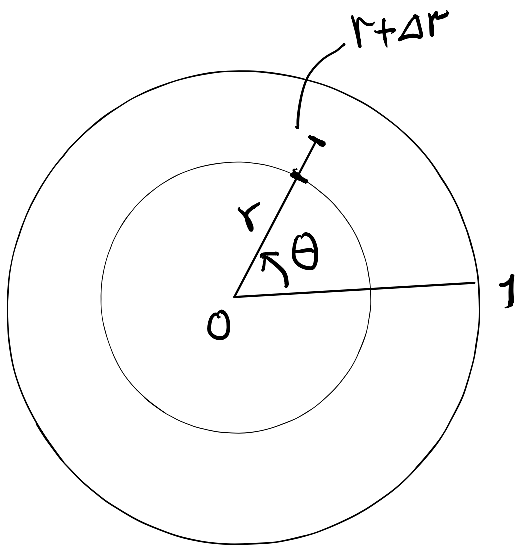
Area in the Disk Model

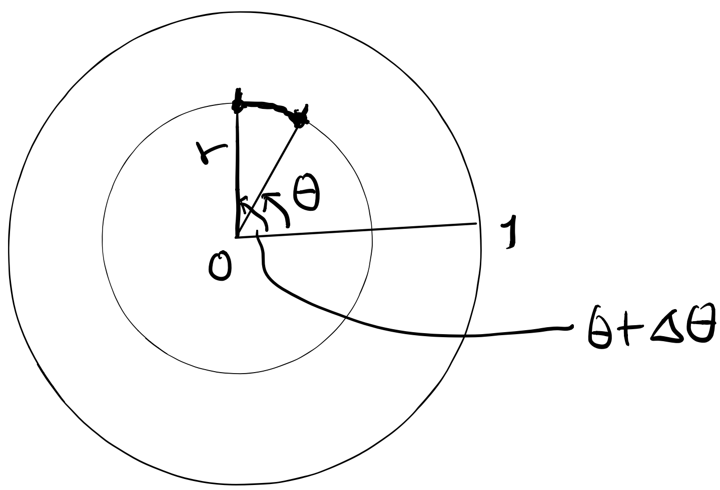


We first calculate the length elements for $\theta = \text{const.}$ and $r = \text{const.}$ in the disk model.

$$z(r) = re^{i\theta} \qquad \theta = \text{fixed}$$

$$\Rightarrow z'(r) = e^{i\theta} \qquad \text{length} = \int_r^{r+\Delta r} \frac{2|z'(r)|}{1-|z(r)|^2}\,dr$$

$$\sim \frac{2}{1-r^2}\Delta r$$

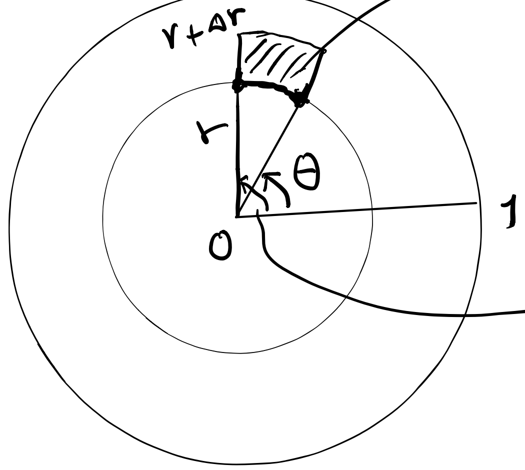


$$z(\theta) = re^{i\theta}, \quad r = \text{fixed}$$

$$z'(\theta) = ire^{i\theta}$$

$$\text{length} = \int_\theta^{\theta+\Delta\theta} \frac{2|z'(\theta)|}{1-|z(\theta)|^2}\,d\theta$$

$$\sim \frac{2r}{1-r^2}\Delta\theta$$



$$\sim \frac{2\Delta r}{1-r^2}\cdot\frac{2r}{1-r^2}\Delta\theta$$

$$\longrightarrow \frac{4r\,dr\,d\theta}{(1-r^2)^2}$$

<u>Def</u>: The <u>area</u> of a region R in the hyperbolic plane (unit disk model) is defined by

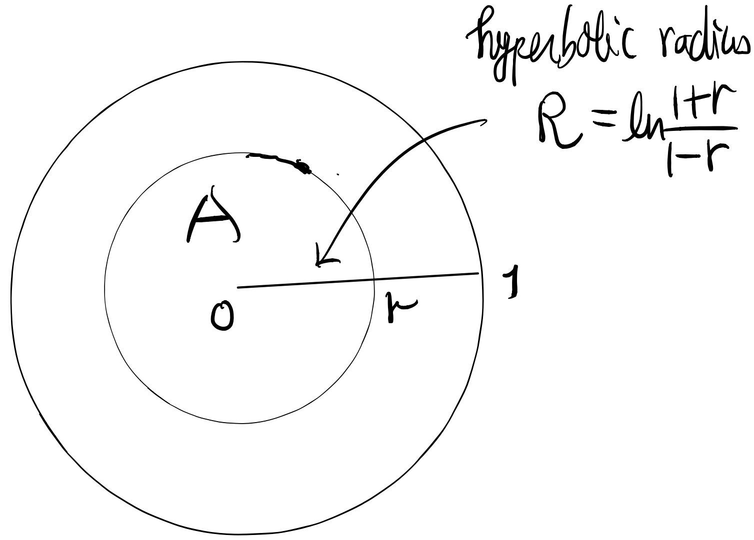
$$\boxed{A = \iint_R \frac{4r}{(1-r^2)^2}\,dr\,d\theta}$$

$$= \iint_R \frac{4}{(1-x^2-y^2)^2}\,dx\,dy$$

<u>eg</u>: Area of hyperbolic circle with hyperbolic radius R $= 4\pi\sinh^2\left(\frac{R}{2}\right)$

<u>Pf</u>: $A = \int_0^{2\pi}\int_0^{r} \frac{4r}{(1-r^2)^2}\,dr\,d\theta$

$$= 4\pi \int_0^r \frac{2r\,dr}{(1-r^2)^2}$$



= integrate to get a formula in r

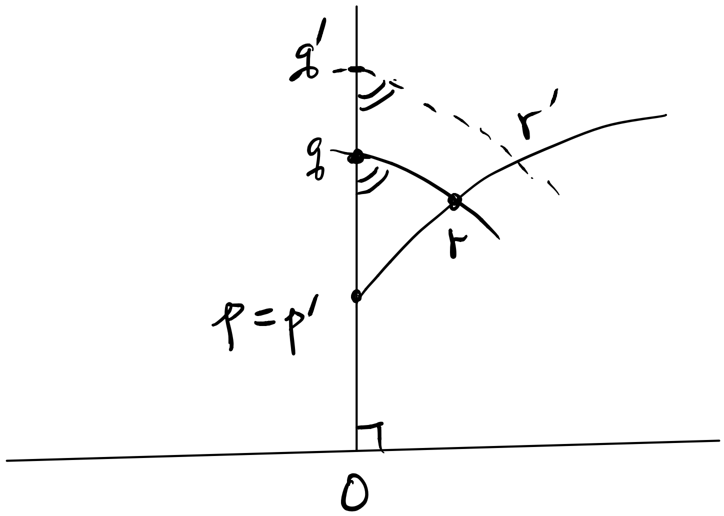
(optional Ex!)

= use $R = \ln\frac{1+r}{1-r}$ to get the required formula $4\pi \sinh^2\frac{R}{2}$. ✕

Similarity

Thm: If corresponding angles are equal in 2 (hyperbolic) triangles $\triangle pqr$ & $\triangle p'q'r'$, then the hyperbolic triangles are congruent.

Proof: In upper half-plane model, we may put $p = p'$ and $\overline{pq}$ & $\overline{p'q'}$ along the y-axis and both q, q' above p.



If $q \neq q'$, by a scaling, which is a transformation in the hyperbolic group, the hyperbolic straight line containing $\overline{qr}$ transforms to a hyperbolic straight line passing through the point q', which makes an angle equal to

$\angle pqr = \angle p'q'r'$

$\Rightarrow$ r' is on this hyperbolic straight line (by assumption)

$\Rightarrow$ r' is the intersection point of this hyperbolic straight line and $\overline{pr}$.

This implies $A(\Delta pqr) \neq A(\Delta p'q'r')$ which is a contradiction since both areas equal to

$\pi -$ (sum of interior angles).

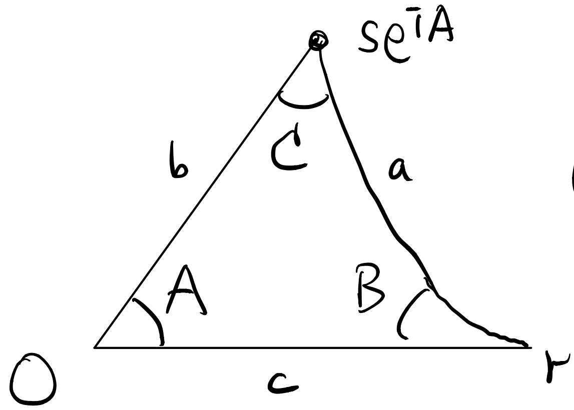
$\therefore\ q = q'$, ie. $\overline{pq} \cong \overline{p'q'}$

Similarly $\overline{pr} \cong \overline{p'r'}$ & $\overline{qr} \cong \overline{q'r'}$.

Hence $\Delta pqr \cong \Delta p'q'r'$. ※

Cosine rule I (in hyperbolic geometry)

disk model



$(0 < r, s < 1)$

Then $$\boxed{\operatorname{ch} a = \operatorname{ch} b \operatorname{ch} c - \operatorname{sh} b \operatorname{sh} c \cos A}$$

where $\begin{cases} \operatorname{ch} x = \cosh x = \dfrac{e^x + e^{-x}}{2} \\ \operatorname{sh} x = \sinh x = \dfrac{e^x - e^{-x}}{2} \end{cases}$

Cosine Rule II

$$\boxed{\operatorname{ch} a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}}$$

Sine Rule

$$\boxed{\frac{\sin A}{\operatorname{sh} a} = \frac{\sin B}{\operatorname{sh} b} = \frac{\sin C}{\operatorname{sh} c}}$$

Pf of Cosine Rule I

By our notation, we have

$$|r - se^{iA}|^2 = r^2 + s^2 - 2rs\cos A$$

In hyperbolic geometry

$$\begin{cases} c = \ln \frac{1+r}{1-r}, \quad b = \ln \frac{1+s}{1-s}, \text{ and} \\ a = \ln \dfrac{1 + \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right|}{1 - \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right|} \end{cases}$$

$$\Rightarrow \quad r = \frac{e^c - 1}{e^c + 1} = \frac{\cancel{e^{\frac{c}{2}}}(e^{\frac{c}{2}} - e^{-\frac{c}{2}})/2}{\cancel{e^{\frac{c}{2}}}(e^{\frac{c}{2}} + e^{-\frac{c}{2}})/2} = \frac{sh\frac{c}{2}}{ch\frac{c}{2}} = \tanh\frac{c}{2}$$

Similarly

$$\begin{cases} r = \tanh\frac{c}{2} \\ s = \tanh\frac{b}{2} \\ \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right| = \tanh\frac{a}{2} \end{cases}$$

$$\Rightarrow \quad \tanh^2\frac{a}{2} = \frac{|r - se^{iA}|^2}{|1 - rse^{iA}|^2} = \frac{r^2 - 2rs\cos A + s^2}{1 - 2rs\cos A + r^2s^2}$$

$$\Rightarrow \quad ch\,a = \frac{ch\,a}{1} = \frac{ch^2\frac{a}{2} + sh^2\frac{a}{2}}{ch^2\frac{a}{2} - sh^2\frac{a}{2}} \qquad (\text{Ex!})$$

$$= \frac{1 + \tanh^2\frac{a}{2}}{1 - \tanh^2\frac{a}{2}}$$

$$= \frac{(1 - 2rs\cos A + r^2s^2) + (r^2 - 2rs\cos A + s^2)}{(1 - 2rs\cos A + r^2s^2) - (r^2 - 2rs\cos A + s^2)}$$

$$= \frac{1 + r^2 + s^2 + r^2s^2 - 4rs\cos A}{1 - r^2 - s^2 + r^2s^2}$$

$$= \frac{(1+r^2)(1+s^2) - 4rs\cos A}{(1-r^2)(1-s^2)}$$

$$= \left(\frac{1+r^2}{1-r^2}\right)\left(\frac{1+s^2}{1-s^2}\right) - \left(\frac{2r}{1-r^2}\right)\left(\frac{2s}{1-s^2}\right)\cos A$$

$$\frac{1+r^2}{1-r^2} = \frac{1+\tanh^2\frac{c}{2}}{1-\tanh^2\frac{c}{2}} = \frac{\mathrm{ch}^2\frac{c}{2}+\mathrm{sh}^2\frac{c}{2}}{\mathrm{ch}^2\frac{c}{2}-\mathrm{sh}^2\frac{c}{2}} = \frac{\mathrm{ch}\,c}{1} = \mathrm{ch}\,c$$

Similarly $\frac{1+s^2}{1-s^2} = \mathrm{ch}\,b$

$$\frac{2r}{1-r^2} = \frac{2\tanh\frac{c}{2}}{1-\tanh^2\frac{c}{2}} = \frac{\frac{2\,\mathrm{sh}\frac{c}{2}}{\mathrm{ch}\frac{c}{2}}}{1-\frac{\mathrm{sh}^2\frac{c}{2}}{\mathrm{ch}^2\frac{c}{2}}}$$

$$= \frac{2\,\mathrm{sh}\frac{c}{2}\,\mathrm{ch}\frac{c}{2}}{\mathrm{ch}^2\frac{c}{2}-\mathrm{sh}^2\frac{c}{2}} = \frac{\mathrm{sh}\,c}{1} = \mathrm{sh}\,c \quad (\text{Ex!})$$

Similarly $\frac{2s}{1-s^2} = \mathrm{sh}\,b$.

Hence $\mathrm{ch}\,a = \mathrm{ch}\,b\,\mathrm{ch}\,c - \mathrm{sh}\,b\,\mathrm{sh}\,c\cos A$. #

(Pf of Cosine Rule II is omitted)

Pf of Sine Rule

by Cosine Rule I,

$$\left(\frac{\sin A}{\operatorname{sh} a}\right)^2 = \frac{1-\cos^2 A}{\operatorname{sh}^2 a}$$

$$= \frac{1-\left(\frac{\operatorname{ch} b \operatorname{ch} c - \operatorname{ch} a}{\operatorname{sh} b \operatorname{sh} c}\right)^2}{\operatorname{sh}^2 a}$$

$$= \frac{\operatorname{sh}^2 b \operatorname{sh}^2 c - (\operatorname{ch} b \operatorname{ch} c - \operatorname{ch} a)^2}{\operatorname{sh}^2 a \operatorname{sh}^2 b \operatorname{sh}^2 c}$$

$$= \frac{(\operatorname{ch}^2 b - 1)(\operatorname{ch}^2 c - 1) - (\operatorname{ch}^2 b \operatorname{ch}^2 c - 2\operatorname{ch} a \operatorname{ch} b \operatorname{ch} c + \operatorname{ch}^2 a)}{\operatorname{sh}^2 a \operatorname{sh}^2 b \operatorname{sh}^2 c}$$

$$= \frac{1-(\operatorname{ch}^2 a + \operatorname{ch}^2 b + \operatorname{ch}^2 c) + 2\operatorname{ch} a \operatorname{ch} b \operatorname{ch} c}{\operatorname{sh}^2 a \operatorname{sh}^2 b \operatorname{sh}^2 c}.$$

By symmetry of the RHS in a, b, c, we have

$$\left(\frac{\sin A}{\operatorname{sh} a}\right)^2 = \left(\frac{\sin B}{\operatorname{sh} b}\right)^2 = \left(\frac{\sin C}{\operatorname{sh} c}\right)^2$$

Since $A+B+C < \pi$, $(A, B, C > 0)$

we have $\sin A, \sin B, \sin C > 0$.

Hence $$\frac{\sin A}{\operatorname{sh} a} = \frac{\sin B}{\operatorname{sh} b} = \frac{\sin C}{\operatorname{sh} c}.$$

Ch Quaterion (四元數) (Ch17 of the reference)

Def: A quaternion is a "number" of the form

$$\boxed{a+bi+cj+dk}$$

where $a,b,c,d \in \mathbb{R}$.

i,j,k are square roots of -1.

(ie. $\boxed{i^2=j^2=k^2=-1}$)

in addition: $\boxed{ijk=-1}$

With usual "addition" and "multiplication" laws except the following

$$\begin{cases} ij=-ji=k \\ jk=-kj=i \\ ki=-ik=j \end{cases}$$

Pf: (of $ij=k$):

$$ij=(-ij)(-1)=(-ij)(k^2)$$

$= -(ijk)k = k$ $\quad\square$

egs: (i) $(1+2i+3j+4k)+(2-3i+4j-5k)$

$= (1+2)+(2+(-3))i+(3+4)j+(4+(-5))k$

$= 3-i+7j-k.$

(ii) $(2i+j)(j+k)$

$= (2i+j)j+(2i+j)k$

$= 2ij+j^2+2ik+jk$

$= 2k-1+(-j)+i$

$= -1+i-j+2k.$

Thm Quaternion multiplication has the following properties:

(a) Associativity: $q(rs)=(qr)s$

(b) Distributivity: $q(r+s)=qr+qs$

(c) Inverses: $\forall$ quaterion $q\neq 0$ $(0 \overset{\text{def}}{=} 0+0i+0j+0k)$ $\exists$ a quaterion r st. $qr=1$.

Models for quaternions:

Let consider 4×4 matrices:

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 1 \\ \hline -1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -1 & & \end{array}\right], \quad \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & 1 & 0 \end{array}\right] \quad \& \quad \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & 1 & 0 \\ \hline 0 & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \end{array}\right]$$

Then (for instance)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -I_{4\times 4}$$

Similarly $\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 1 \\ \hline -1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & -1 & & \end{array}\right]^2 = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & -1 \\ & & 1 & 0 \\ \hline 0 & -1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \end{array}\right]^2 = -I_{4\times 4}$ (check!)

Finally

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] = -I_{4\times 4}$$

(Check!)

So we can model quaternions by letting

$$1 \longmapsto \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = I_{4\times 4}$$

$$i \longmapsto \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$j \longmapsto \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$k \longmapsto \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

Then a quaternion

$$q = t + xi + yj + zk$$

$$\mapsto t\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+x\begin{bmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{bmatrix}+y\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}+z\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}t&y&x&-z\\-y&t&z&x\\-x&-z&t&y\\z&-x&-y&t\end{bmatrix}$$

$$\therefore \boxed{\text{Quaternions}=\left\{\begin{bmatrix}t&y&x&-z\\-y&t&z&x\\-x&-z&t&y\\z&-x&-y&t\end{bmatrix}: t,x,y,z\in\mathbb{R}\right\}}$$

with usual matrix addition & multiplication!

$$\left(\text{Compare: } a+bi \longleftrightarrow \begin{bmatrix}a&-b\\b&a\end{bmatrix}\right)$$

Cartersian Form

$$q = t + xi + yj + zk$$

(analogus to $a+bi$ of a complex number)

<u>Scalar part</u> of $q = t + xi + yj + zk$

is defined by $\boxed{Sq = t}$

<u>Vector part</u> by $\boxed{Vq = xi + yj + zk}$

<u>Note</u>: $Sq \in \mathbb{R}$ but Vq is a quaternion.

<u>Conjugate</u> of $q = t + xi + yj + zk$ is defined as

$$\boxed{\begin{aligned} q^* &= Sq - Vq \\ &= t - xi - yj - zk \end{aligned}}$$

<u>Modulus</u>

$$\boxed{|q| \overset{\text{def}}{=} \sqrt{t^2 + x^2 + y^2 + z^2} \overset{\text{Thm}}{=} \sqrt{qq^*}}$$

If $|q| = 1$, q is called a <u>unit quaternion</u>.

If $Sq = 0$, then q is called a <u>pure quaternion</u>.

eg: Every pure, unit quaternion is a square root of -1.

Pf: Let q be a pure unit quaternion

then $q = xi + yj + zk$ with

$$|q|^2 = x^2 + y^2 + z^2 = 1$$

Hence

$$\begin{aligned} q^2 &= (xi+yj+zk)(xi+yj+zk) \\ &= (xi)(xi) + (yj)(xi) + (zk)(xi) \\ &\quad + (xi)(yj) + (yj)(yj) + (zk)(yj) \\ &\quad + (xi)(zk) + (yj)(zk) + (zk)(zk) \\ &= x^2 i^2 + \cancel{(xy)(ji)} + \cancel{(zx)(ki)} \\ &\quad + \cancel{(xy)(ij)} + y^2 j^2 + \cancel{(zy)(kj)} \\ &\quad + \cancel{(xz)(ik)} + \cancel{(yz)(jk)} + z^2 k^2 \\ &= -x^2 - y^2 - z^2 = -1 \end{aligned}$$

since $ij = -ji$, $ki = -ik$, $kj = -jk$, &

$$i^2 = j^2 = k^2 = -1$$

#

Note: in fact, we've proved that for <u>pure quaternions</u>

$$q^2 = -|q|^2 .$$

Pure Quaternions as vectors in $\mathbb{R}^3$

If $q = x_1 i + y_1 j + z_1 k$

$r = x_2 i + y_2 j + z_2 k$

then
$$qr = (x_1 i + y_1 j + z_1 k)(x_2 i + y_2 j + z_2 k)$$

$$= -x_1x_2 + y_1x_2(ji) + z_1x_2(ki)$$
$$+ x_1y_2(ij) - y_1y_2 + z_1y_2(kj)$$
$$+ x_1z_2(ik) + y_1z_2(jk) - z_1z_2$$

$$= -x_1x_2 - y_1x_2k + z_1x_2j$$
$$+ x_1y_2k - y_1y_2 - z y_2 i$$
$$- x_1z_2j + y_1z_2i - z_1z_2$$

$$= -(x_1x_2 + y_1y_2 + z_1z_2)$$
$$+ (y_1z_2 - z_1y_2)i - (x_1z_2 - z_1x_2)j + (x_1y_2 - y_1x_2)k$$

$\therefore$ $$\boxed{\begin{aligned} -S(qr) &= x_1x_2 + y_1y_2 + z_1z_2 \\ &= q \cdot r \quad \text{the dot product of } q \,\&\, r \text{ as 3-vectors.} \end{aligned}}$$

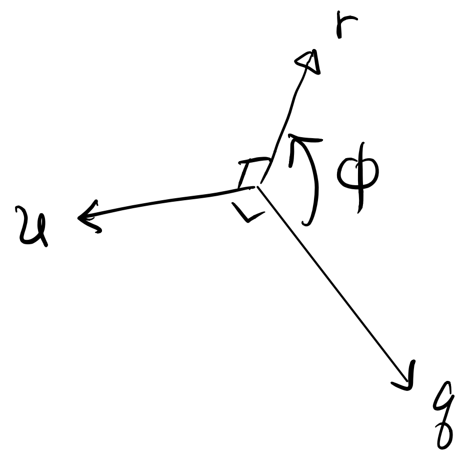
and

$$\boxed{\begin{aligned} V(qr) &= (y_1z_2 - z_1y_2)\,i - (x_1z_2 - z_1x_2)\,j + (x_1y_2 - y_1x_2)\,k \\ &= q \times r \quad \text{the cross-product of } q \,\&\, r \text{ as 3-vectors} \end{aligned}}$$

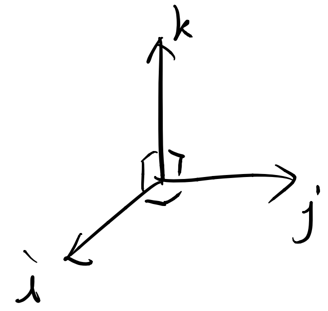
This shows that

$$\boxed{\begin{aligned} -S(qr) &= q \cdot r = |q||r|\cos\phi \\ V(qr) &= q \times r = (|q||r|\sin\phi)\,u \end{aligned}}$$

where
- ϕ = angle between the vectors q & r
- u is a pure unit quaternion representing a unit vector in $\mathbb{R}^3$ perpendicular to q and r, such that (q, r, u) forms a right-handed system



explicit eg: $i \times j = k$



Note: We've used the fact that

modulus of q as pure quaternion

= modulus of q as a 3-vector.

All together, we have for pure quaternions q & r

$$\boxed{qr = -(q \cdot r) + q \times r}$$

$\uparrow$ quaternion multiplication

$\uparrow$ dot product

$\uparrow$ cross product.

Polar form

Thm: Every quaternion can be represented in the form

$$q = |q| (\cos\theta + u \sin\theta)$$

where $\theta \in \mathbb{R}$; u is a pure unit quaternion

$$(u^2 = -1)$$

Remark:

complex	quaternion
$z = \lvert z \rvert (\cos\theta + i \sin\theta)$	$q = \lvert q \rvert (\cos\theta + u \sin\theta)$
$(\pm i)^2 = -1$ (0-dim'l)	$u^2 = -1$ (2-dim'l)

Pf: Let $q = t + xi + yj + zk$

Set $u = \dfrac{xi + yj + zk}{r}$, where $r = \sqrt{x^2 + y^2 + z^2}$

Then u is a pure unit quaternion

and
$$q = t + ru$$
$$= |q| \left(\frac{t}{|q|} + \frac{r}{|q|} u \right)$$

Note $\left(\frac{t}{|q|}\right)^2 + \left(\frac{r}{|q|}\right)^2 = \frac{t^2+(x^2+y^2+z^2)}{|q|^2} = 1$

$\Rightarrow \exists \theta \in \mathbb{R}$ s.t. $\quad \frac{t}{|q|} = \cos\theta$ & $\frac{r}{|q|} = \sin\theta$

$\therefore \quad q = |q|(\cos\theta + u\sin\theta)$ ※