Normal Form of a Möbius Transformation

Let T = Möb transformation with 2 fixed points p & q.

Since (a) T fixes p & q and

(b) T maps clines to clines,

T maps clines passing thro. p & q to another clines passing thro. p & q.

$\therefore$ Steiner circles of 1st kind wrt p & q are invariant under T.

$\Rightarrow$ One can show that Steiner circles of 2nd kind are also invariant under T (Ex!)

To see the action of T, we consider again

$$w = Sz = \frac{z-p}{z-q}$$

$$\begin{tikzcd} w \in \hat{\mathbb{C}} \arrow[r, "R"] \arrow[d, "S^{-1}"'] & \hat{\mathbb{C}} \arrow[d, "S^{-1}"] \\ z \in \hat{\mathbb{C}} \arrow[r, "T"] & \hat{\mathbb{C}} \end{tikzcd} \qquad \left(\begin{tikzcd} z \in \hat{\mathbb{C}} \arrow[r, "T"] \arrow[d, "S"'] & \hat{\mathbb{C}} \arrow[d, "S"] \\ w \in \hat{\mathbb{C}} \arrow[r] & \hat{\mathbb{C}} \end{tikzcd} \right)$$

Let $R(w) = STS^{-1}(w)$ be the lift of T to the (extended) w-plane via S^{-1}.

Then $\begin{cases} R(0) = STS^{-1}(0) = ST(p) = S(p) = 0 \\ R(\infty) = STS^{-1}(\infty) = ST(q) = S(q) = \infty. \end{cases}$

One the other hand, R is of the form

$$Rw = \frac{aw+b}{cw+d}, \text{ for some } a,b,c,d \in \mathbb{C} \text{ with } ad-bc \neq 0.$$

Then $\begin{cases} R(0) = 0 \Rightarrow b = 0 \\ R(\infty) = \infty \Rightarrow c = 0 \end{cases}$

$\therefore\ Rw = \left(\frac{a}{d}\right) w \quad \left(a, d \neq 0 \text{ since } 0 \neq ad - bc = ad \right)$

Write $\lambda = \frac{a}{d} \neq 0$, we have

$$\boxed{Rw = \lambda w}$$

Substituting into $Rw = STS^{-1}w$,

$\Rightarrow$ $$\lambda w = STS^{-1}w$$

$\Rightarrow$ $$\lambda(Sz) = STS^{-1}(Sz) = S(Tz)$$

$$\boxed{\lambda \frac{z-p}{z-q} = \frac{Tz-p}{Tz-q}} \quad (\lambda \neq 0)$$

is called the normal form of T.

We see that T can be understood as composition of 3 operations:

(i) sending the fixed points to 0 and ∞.

(ii) multiplication by a nonzero complex constant $\lambda \neq 0$.

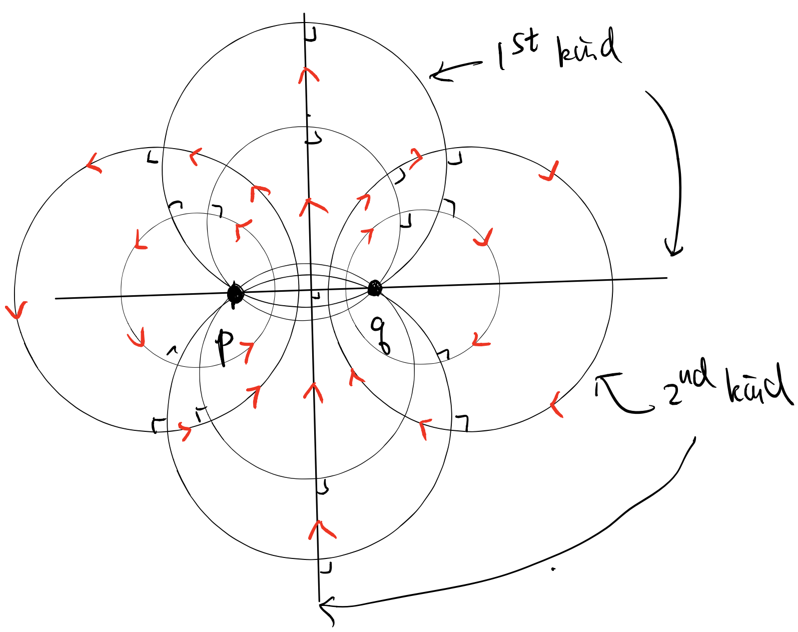
(iii) sending 0 and ∞ back to the fixed points.

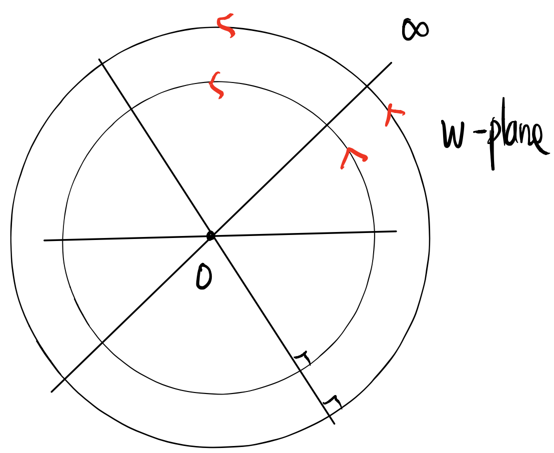
Case 1 Elliptic transformation ($|\lambda| = 1$)

$\lambda = e^{i\theta} \Rightarrow Rw = e^{i\theta} w$ is a rotation about the origin

$\Rightarrow$ the action of T is to move points along the Steiner circles of 2nd kind "around the fixed points".

(Moreover, T sends Steiner circles of 1st kind to (another) Steiner circles of 1st kind.)



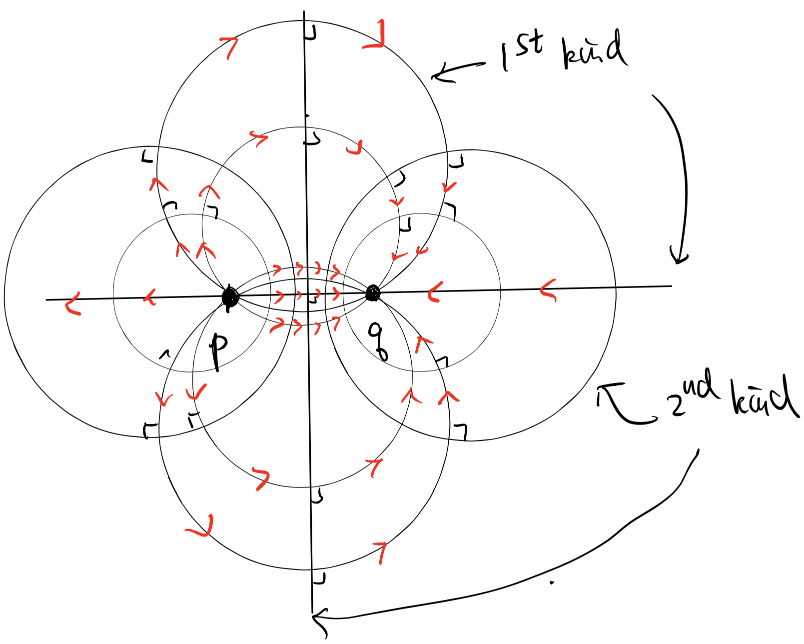


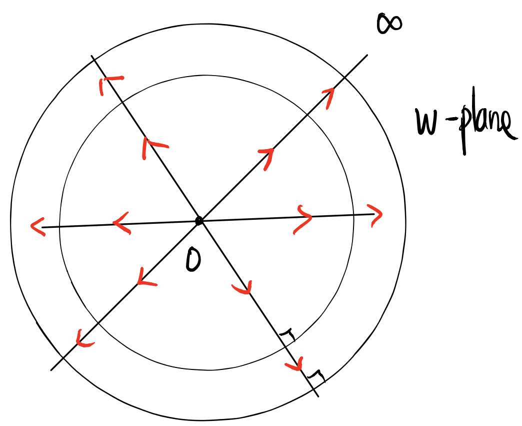
Case 2 Hyperbolic transformation ($\lambda > 0$)

$Rw = \lambda w$, $\lambda > 0$ is a homothetic transformation

$\Rightarrow$ the action of T is to move points along the Steiner circle of the 1st kind

(Moreover, T sends Steiner circles of the 2nd kind to (another) Steiner circles of the 2nd kind.)





Case 3 Loxodromic Transformation

$\lambda = ke^{i\theta}$, $k \neq 1$, $k > 0$, and $\theta \neq 0 \pmod{2\pi}$

action of T = a combination of the motions of an elliptic and a hyperbolic transformation.

Conclusions

(1) 2 kinds of Steiner circles
$\rightarrow$ generalized polar coordinates for Möbius Geometry.

(2) Möbius transformations with 2 fixed points transform each Steiner circle wrt the fixed points to (itself or another) Steiner circle (of the same kind) wrt the same fixed point.

(3) Simplest typers of transformations with 2 fixed points:
(a) Elliptic (rotation): move points along Steiner circles of 2^{nd} kind.
(b) Hyperbolic (scaling): move points along Steiner circles of 1^{st} kind.

(4) Loxodromic = Combination of elliptic & hyperbolic

(5) Normal form = expression of the relationship between the transformation and the Steiner circle coordinate system determined by its fixed points.

Parabolic Transformation (Transformation with 1 fixed point)

Let T be a transformation with one fixed point p.

Consider $w = Sz = \frac{1}{z-p}$.

Then $S(p) = \infty$

$$\begin{tikzcd} w \in \hat{\mathbb{C}} \arrow[r, "R"] \arrow[d, "S^{-1}"'] & \hat{\mathbb{C}} \arrow[d, "S^{-1}"] \\ z \in \hat{\mathbb{C}} \arrow[r, "T"] & \hat{\mathbb{C}} \end{tikzcd}$$

And $R = STS^{-1}$

satisfies

$$R(\infty) = STS^{-1}(\infty)$$
$$= ST(p) = Sp = \infty$$

Using the form $Rw = \frac{aw+b}{cw+d}$, $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$,

we have

$$R(\infty) = \infty \Rightarrow c = 0 \quad (\Rightarrow d \neq 0, a \neq 0)$$

Hence $$Rw = \left(\frac{a}{d}\right) w + \left(\frac{b}{d}\right)$$

Since R has no other fixed point (otherwise T will have 2 fixed points.)

$\Rightarrow$ $w = \left(\frac{a}{d}\right) w + \frac{b}{d}$ has no solution in $\mathbb{C}$

$$\Rightarrow \quad \frac{a}{d} = 1 .$$

$\therefore \quad Rw = w + \beta$ for some $\beta \in \mathbb{C}$,

which is a translation.

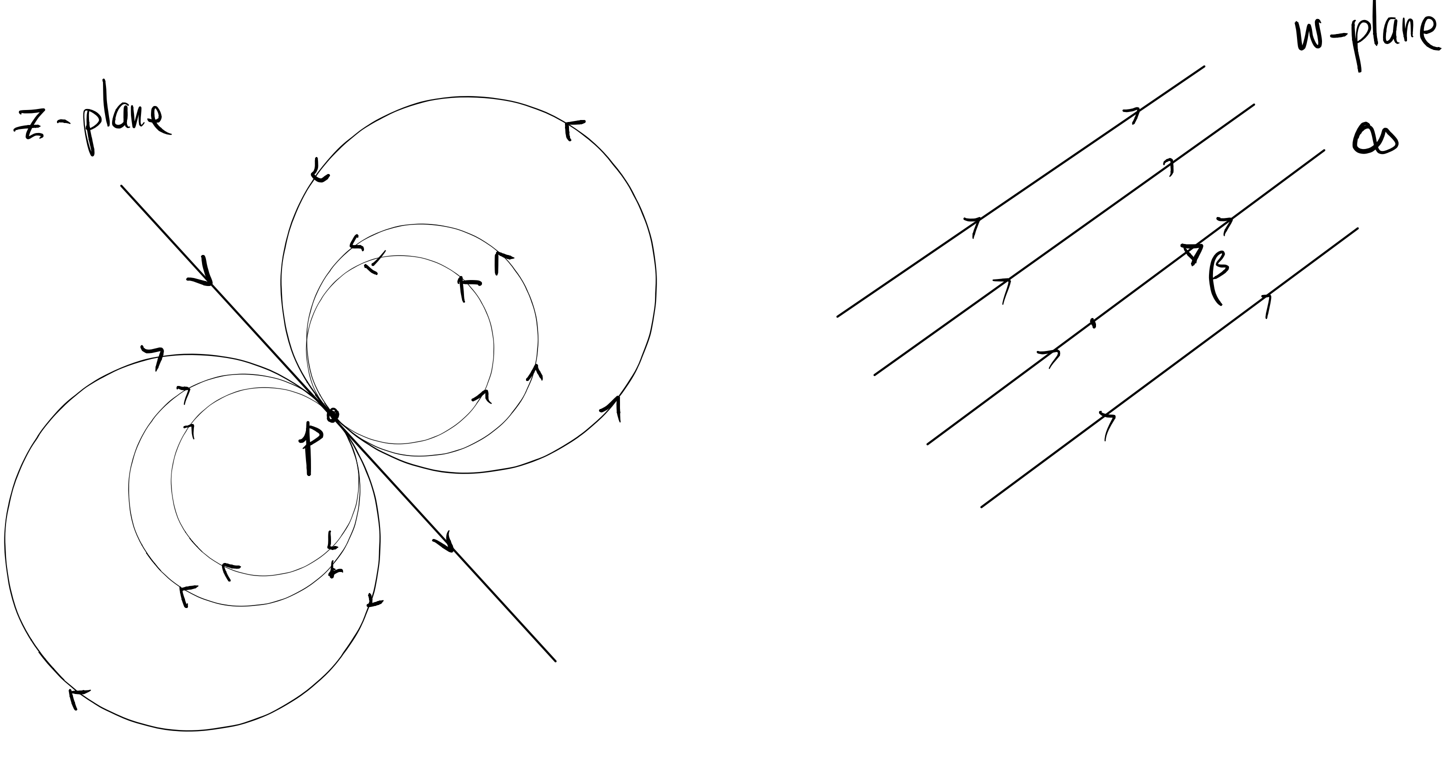
Hence
$$STS^{-1}w = Rw = w + \beta$$

$$\Rightarrow \quad STz = Sz + \beta$$

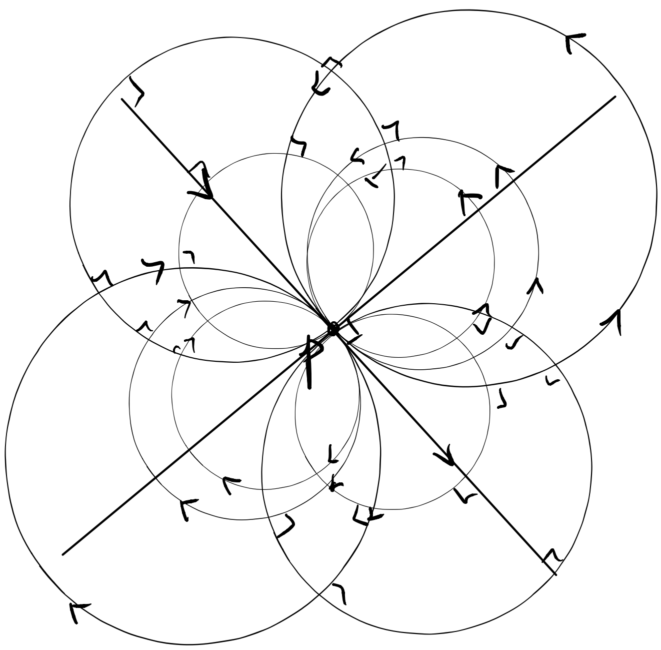
$$\Rightarrow \quad \boxed{\frac{1}{Tz - p} = \frac{1}{z - p} + \beta} \quad (\beta \neq 0)$$

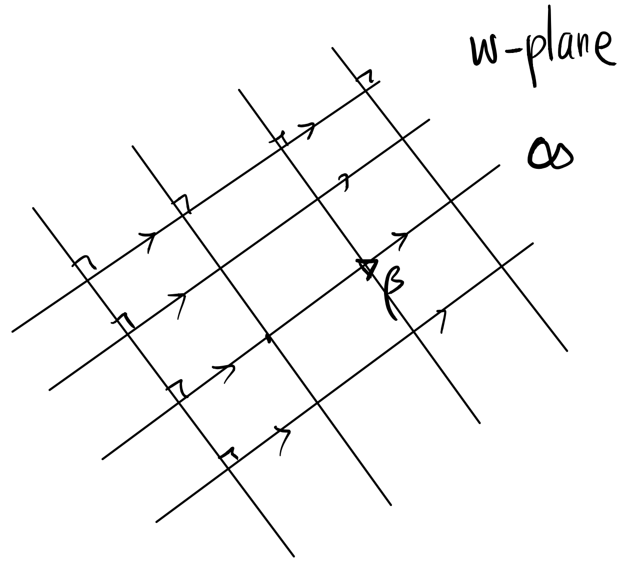
is the <u>normal form</u> of a parabolic transformation

<u>Note</u>: $Rw = w + \beta$ moves point along straight lines parallel to the vector β.



Adding the family of lines orthogonal to line parallel to β, we have a coordinate system on w-plane which gives a coordinate system on the z-plane called a generalized Cartesian coordinate system.





Ch7 Hyperbolic Geometry

This is the non-Euclidean geometry discovered by Gauss, Bolyai, and Lobatchevsky.

There are 2 models of hyperbolic geometry to be discussed in this course:

disk model and

upper half-plane model.

Remark: • Unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$

• Upper half-plane $\mathbb{U} = \{ z \in \mathbb{C} : z = x + iy, y > 0 \}$

Disk model:

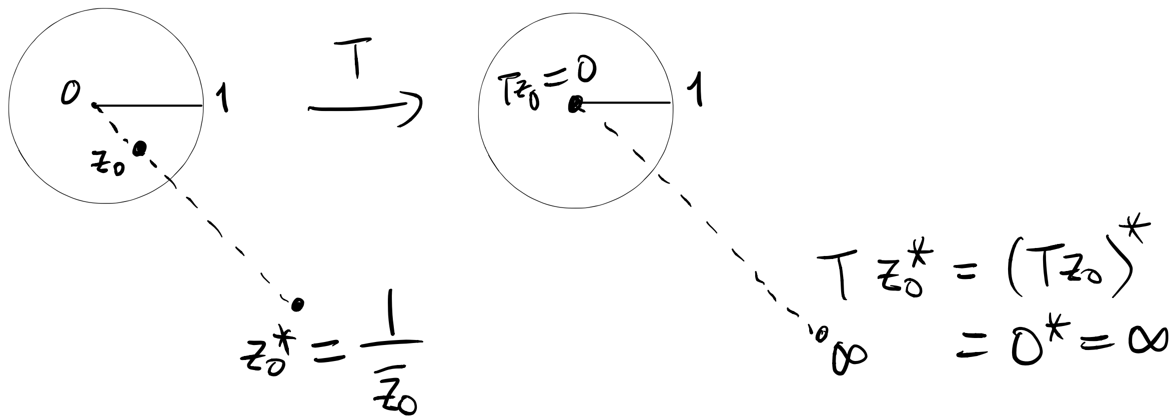
The group of transformations consists of all Möbius transformations that map $\mathbb{D}$ onto itself.

- It is clear that these transformations form a transformation group with underlying space $\mathbb{D}$. (Ex!)
- To find this group explicitly, we let $T \in$ Möb mapping $\mathbb{D}$ onto itself.

Then $|z|<1 \iff |Tz|<1$

And $\exists\ z_0 \in \mathbb{D}$ such that

$$Tz_0 = 0$$



Therefore T mapping $\mathbb{D}$ onto itself

$$\Rightarrow \quad T(z_0^*) = \infty$$

i.e. $\quad T\left(\frac{1}{\bar{z}_0}\right) = \infty$

Hence, we have

$$Tz = \alpha \frac{z - z_0}{z - \frac{1}{\bar{z}_0}} \quad \text{for some } \alpha \in \mathbb{C}\setminus\{0\}$$

$$= (-\alpha \bar{z}_0) \cdot \frac{z - z_0}{1 - \bar{z}_0 z}$$

$$= \lambda \frac{z - z_0}{1 - \bar{z}_0 z} \quad \text{where } \lambda = -\alpha \bar{z}_0$$

Suppose now $|z|=1$, then $|Tz|=1$

$$\therefore \quad 1=|Tz|=\left|\lambda \frac{z-z_0}{1-\bar{z}_0 z}\right|$$

$$=|\lambda| \frac{|z-z_0|}{|1-\bar{z}_0 z|}$$

$$=|\lambda| \frac{|z-z_0|}{|\bar{z} z-\bar{z}_0 z|}$$

$$=\frac{|\lambda|}{|z|} \cdot \frac{|z-z_0|}{|\bar{z}-\bar{z}_0|}=|\lambda|$$

$\Rightarrow \quad \lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

Hence we

<u>Def</u>: Let $\mathbb{D}$ be the unit disk in the complex plane.

- Let $\mathbb{H}$ be the set of transformations of $\mathbb{D}$ of the form $Tz = e^{i\theta}\dfrac{z-z_0}{1-\bar{z}_0 z}$, where $|z_0|<1$, $\theta \in \mathbb{R}$.
- The pair $(\mathbb{D}, \mathbb{H})$ <u>models hyperbolic geometry</u>.
- The set $\mathbb{D}$ will be called the <u>hyperbolic plane</u>.
- The group $\mathbb{H}$ is the <u>hyperbolic group</u>.

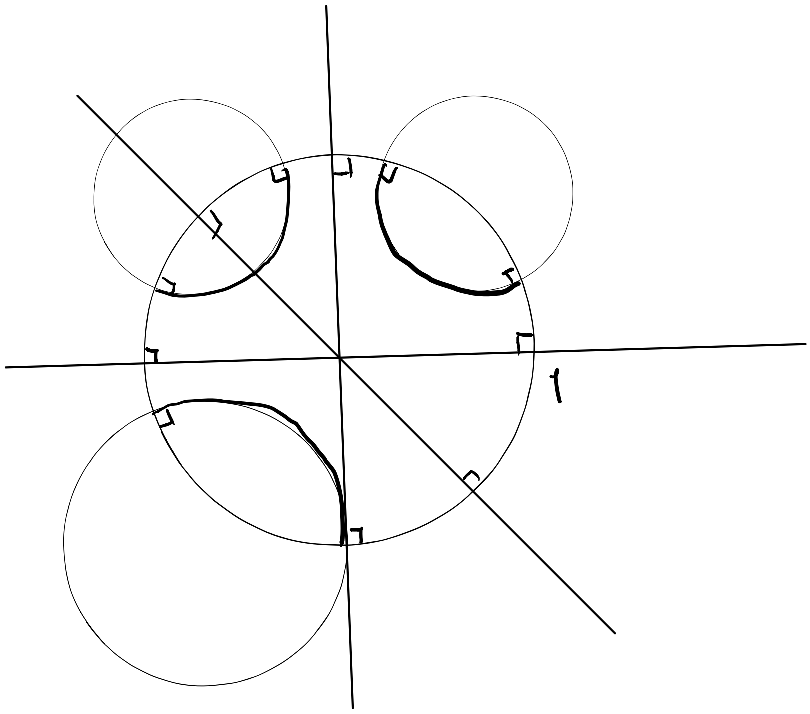
<u>Note</u>: $\mathbb{H}$ is a "subgroup" of the Möbius group $\mathbb{M}$ and $\mathbb{D} \subset \hat{\mathbb{C}}$.

$\Rightarrow$ hyperbolic geometry is a "<u>subgeometry</u>" of Möbius geometry.

Hence: "Every" statement true in Möb geometry is also "true" in hyperbolic geometry!

<u>(Hyperbolic) Straight lines</u>

<u>Def</u> A (<u>hyperbolic</u>) <u>straight line</u> is (the part inside the unit disk) a Euclidean circle or Euclidean straight line in the complex plane that <u>intersects</u> the unit circle at a <u>right angle</u>.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