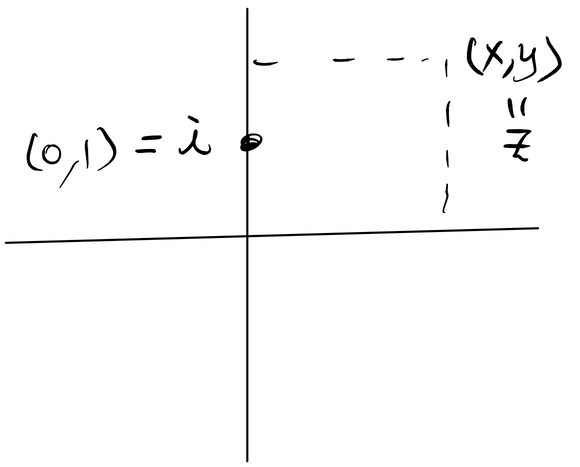
Ch2 Complex Numbers

Def: (i) A <u>complex number</u> is a point $z=(x,y)$ in the Cartesian plane $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$
$= \{(x,y) : x,y \in \mathbb{R}\}$

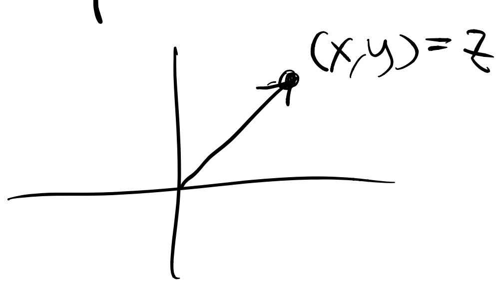


(ii) The complex number $(0,1)$ will be <u>denoted by i</u>, which is called the <u>imaginary unit</u>.

(iii) $x = \mathrm{Re}\, z = \mathrm{Re}(z)$ is called the <u>real part</u> of z,
$y = \mathrm{Im}\, z = \mathrm{Im}(z)$ is called the <u>imaginary part</u> of z.

Operations on complex numbers

(Recall: points in $\mathbb{R}^2 \longleftrightarrow$ 2-vectors starting from $(0,0)$)

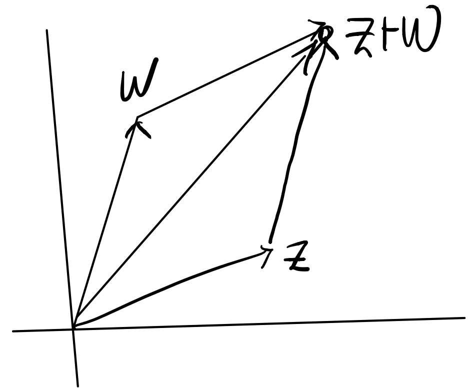


(a) Vector addition $\longleftrightarrow$ addition of complex number

$z = (x, y)$

$w = (s, t)$

$\Rightarrow z + w = (x+s, y+t)$



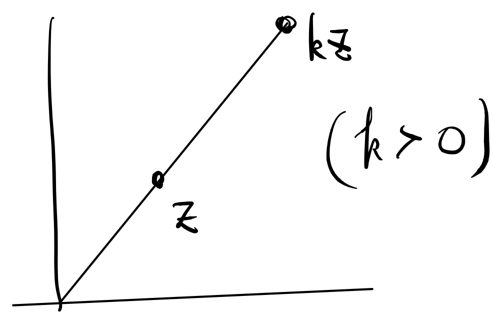
(b) Scalar multiplication of vector

$\longleftrightarrow$ multiplication of a cpx number by a real number

$z = (x, y)$,

$k \in \mathbb{R}$

$\Rightarrow kz = (kx, ky)$



Cartesian form of a cpx number

For $z = (x, y) = x(1,0) + y(0,1)$, it is natural to denote $(1,0)$ simply by 1 and write

$$\boxed{z = x + yi}$$

which is called the Cartesian form of $z = (x, y)$.

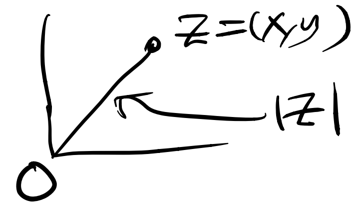
Related Geometric Notions

Algebra	Geometry

(1) Modulus

$|z| = \sqrt{x^2+y^2}$

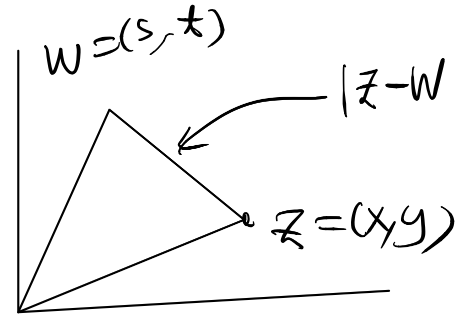
Distance to the origin



(2)

$|z-w| = \sqrt{(x-s)^2+(y-t)^2}$

Distance between 2 points

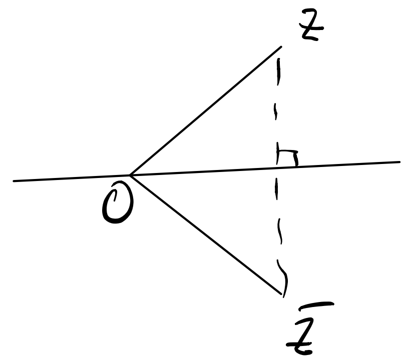


(3) Conjugate

If $z = x + iy$

then $\bar{z} = x - iy$

Reflection across the x-axis



Remark:

(4) $\mathrm{Re}(z\bar{w})$

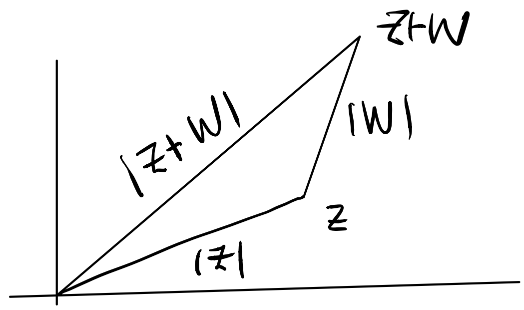
(not yet defined)

Inner product (dot) $z \cdot w$

Remark: Properties of modulus:

(a) Homogeneity: $|kz| = |k||z|$, $\forall k \in \mathbb{R}$ and z cpx.

(b) Triangle inequality $|z+w| \leq |z| + |w|$



Cpx Multiplication

Def: Let $z = x+iy$ & $w = s+it$.

The product of z & w is defined by

$$zw = (x+iy)(s+it) \overset{\text{def}}{=} (xs - yt) + i(ys + xt)$$

Note: This definition follows from the requirements: $i^2 = -1$, distribution law, commutative law and associative law.

<u>Facts</u>: (i) $(zw)u = z(wu)$ associative law

(ii) Let $0=(0,0)$. Then $\forall z \neq 0$, there exists a cpx number denoted by $\frac{1}{z}$ such that

$$z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$$

<u>Polar form</u>: $z = x+iy = r(\cos\theta + i\sin\theta)$

$$= |z| e^{i\theta}$$

where (r,θ) = polar coordinates for $(x,y) \in \mathbb{R}^2$

(θ is not defined for $z=0$)

<u>Note</u>: We've used the Euler formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

(You may simply regard $e^{i\theta}$ as a short form for $\cos\theta + i\sin\theta$)

In general, we have

$$\boxed{\underline{\text{Def}}: \quad e^{z} = e^{x+iy} \overset{\text{def}}{=} e^{x}(\cos y + i\sin y)}$$

Fact: $\boxed{e^{z+w} = e^{z} e^{w}}$

In particular $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}$

(compound angle formula)

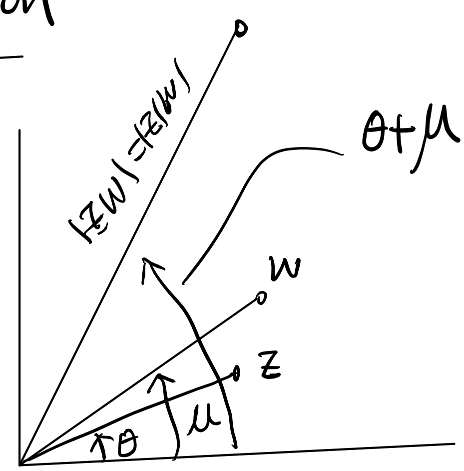
Def: Let $z = re^{i\theta}$ $(z \neq 0)$. Then the <u>argument</u> of z is

$$\boxed{\arg z = \arg(z) = \theta \pmod{2\pi}}$$

The Geometry of cpx multiplication

If $z = |z|e^{i\theta}$, $w = |w|e^{i\mu}$

then $zw = |z||w| e^{i(\theta+\mu)}$



$\therefore \begin{cases} |zw| = |z||w| \\ \arg(zw) = \arg(z) + \arg(w) \pmod{2\pi} \end{cases}$

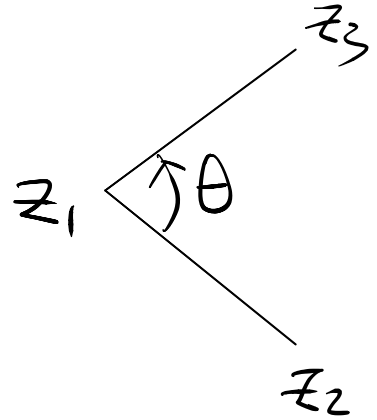
$\therefore$ cpx multiplication = (i) <u>scaling</u>, follows by a (ii) <u>rotation</u>.

when we regard the result of the operation with fixed z:

operator: $$\begin{array}{ccc} \mathbb{C} & \longmapsto & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ w & & zw \end{array}$$

multiplying by z.

<u>Remark</u> :



$$\theta = \arg \frac{z_3 - z_1}{z_2 - z_1} \pmod{2\pi}$$

Ch3 Geometric Transformation (of the plane)

Def: A transformation is a one-to-one (onto) function (mapping) whose image and domain are the same set.

eg: $f: \mathbb{C} \to \mathbb{C}$ is a transformation

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ z & \mapsto & z + (1+2i) \end{array}$$

Pf: (i) one-to-one (injective)

(i.e. if $z_1, z_2 \in \mathbb{C}$ with $f(z_1) = f(z_2)$ then $z_1 = z_2$)

If $f(z_1) = f(z_2)$,

then $z_1 + (1+2i) = z_2 + (1+2i)$

$\Rightarrow$ $z_1 = z_2$ $\therefore$ f is 1-1.

(ii) f is onto (surjective)

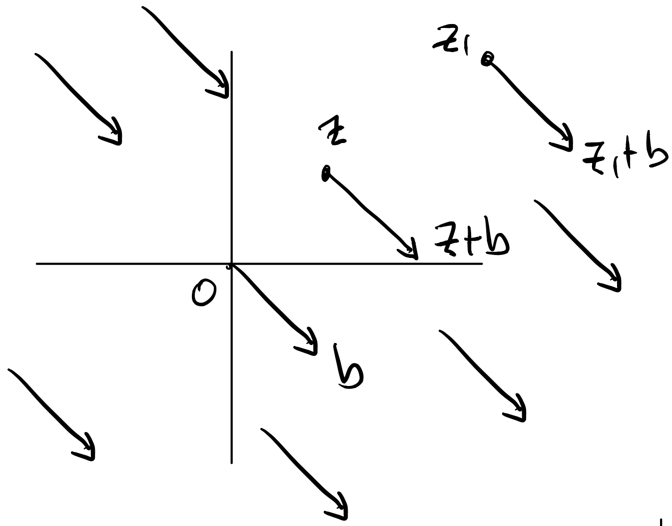
(i.e. $\forall\, w \in \mathbb{C}$, $\exists\, z \in \mathbb{C}$ such that $f(z) = w$)

$\forall\, w \in \mathbb{C}$ $f(\underbrace{w - (1+2i)}_{= z}) = w$ $\therefore f$ is onto #

More generally, $\forall\, b \in \mathbb{C}$, the transformation

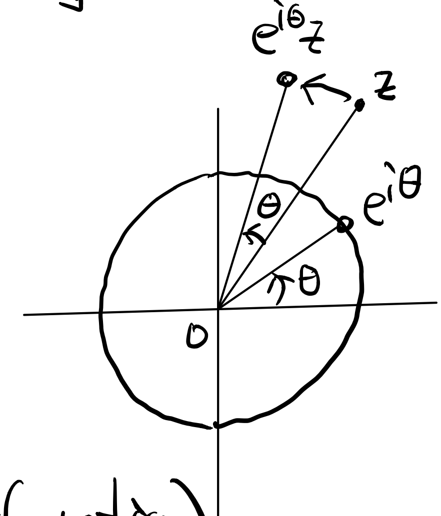
$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ z & \longmapsto & z+b \end{array}$$

is called a <u>translation</u>



<u>eg</u>: <u>Rotation</u>:

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ z & \mapsto & e^{i\theta} z \end{array}$$



(Ex: check that this is a transformation)

<u>eg</u>: <u>Homothetic transformation</u> (stretching or shrinking)

(<u>Scaling</u>)

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ z & \mapsto & kz \end{array}, \quad k \in \mathbb{R}^+ = \{k > 0\}$$

<u>eg</u> : (Cpx) <u>Inversion</u> ($w = \frac{1}{z}$)

$$T : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$$
$$z \mapsto \frac{1}{z}$$

is a transformation (check!)

(i) Points inside the unit circle are transformed to points outside the circle.

<u>Pf</u> : If z = pt. inside the unit circle

then $|z| < 1$

$$\Rightarrow w = \frac{1}{z} = \frac{1}{|z|e^{i\theta}} \quad \text{where } \theta = \arg z$$

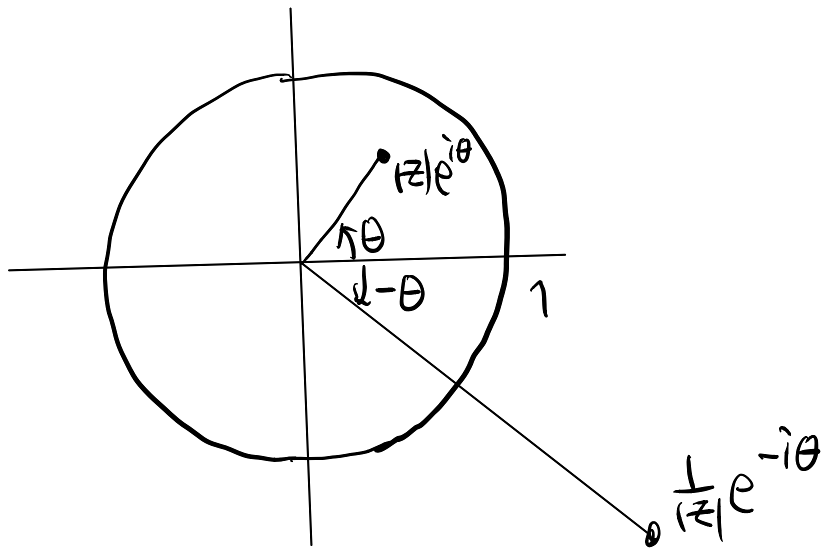
$$= \frac{1}{|z|} e^{-i\theta}$$

$$\Rightarrow |w| = \frac{1}{|z|} > 1$$

$\therefore$ w is outside the unit circle. #

(ii) Similarly, pts outside the unit circle are transformed to pts. inside the unit circle.

(iii) Points in upper half plane exchange places with points in the lower half plane.



(iv) Inversion transforms <u>straight lines</u> to <u>straight lines</u> or <u>circles</u>. More precisely, it transforms a straight line passing thro. 0 to a straight line thro 0; a straight line <u>not</u> passing thro 0 to a circle.

Pf: Let the eqt. of the straight line be

$$ax+by+c=0$$

where a, b, c are real constants (a, b not both zero)

Denote $w = s+it = \frac{1}{z} = \frac{1}{x+iy}$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

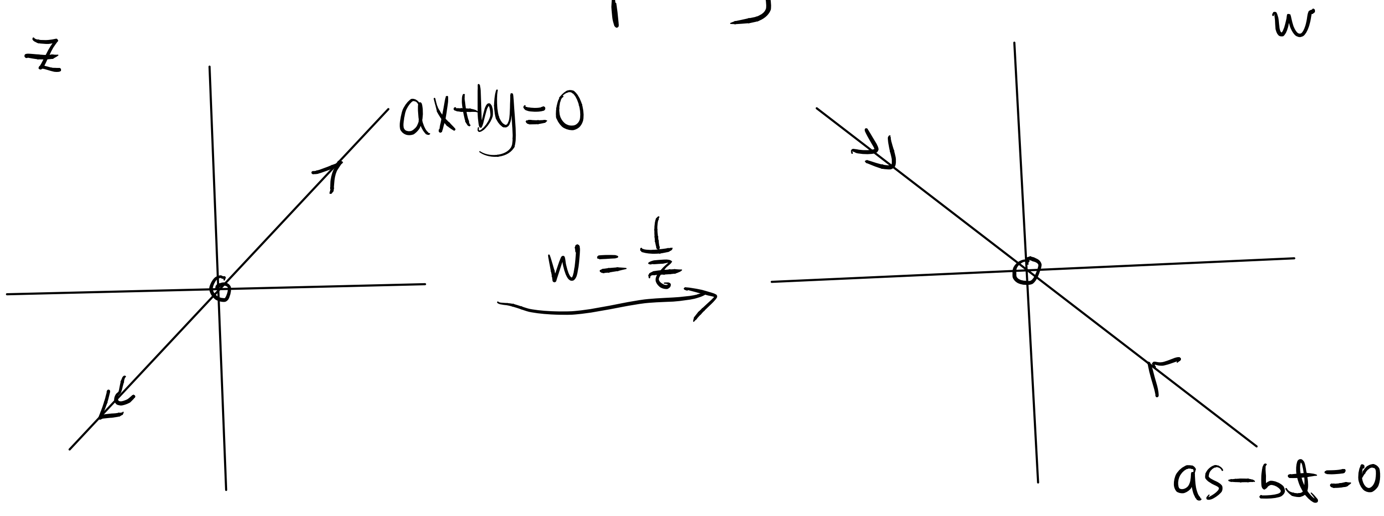
i.e. $\begin{cases} s = \dfrac{x}{x^2+y^2} \\ t = \dfrac{-y}{x^2+y^2} \end{cases}$

Then $\begin{cases} \text{(i)} \quad s^2+t^2 = \dfrac{1}{x^2+y^2} \quad \left(|w|^2 = \dfrac{1}{|z|^2}\right) \\ \text{(ii)} \quad as-bt = -c(s^2+t^2) \quad (\text{check!}) \end{cases}$

<u>Case 1</u>: If $ax+by+c=0$ is a straight line passing thro O, then $c=0$

$\Rightarrow \quad as-bt=0$

$\therefore \quad w=(s,t)$ belongs to a straight line passing thro. O.



Case 2 $c \neq 0$ (straight line <u>not</u> passing thro. 0)

Then $s^2+t^2+\left(\frac{a}{c}\right)s-\left(\frac{b}{c}\right)t=0$

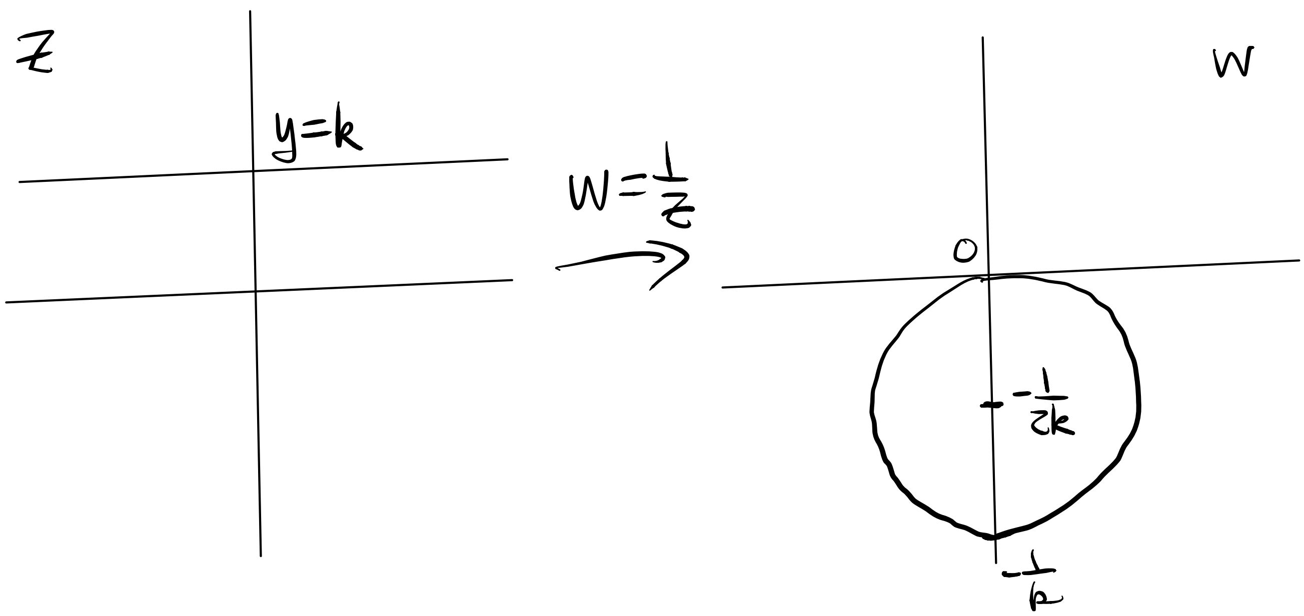
$$\Rightarrow \left(s+\frac{a}{2c}\right)^2+\left(t-\frac{b}{2c}\right)^2=\frac{a^2+b^2}{4c^2}=\left(\frac{\sqrt{a^2+b^2}}{2|c|}\right)^2>0$$

$\therefore$ $w=(s,t)$ belongs to the circle centered at $\left(-\frac{a}{2c},\frac{b}{2c}\right)$ with radius $\frac{\sqrt{a^2+b^2}}{2|c|}$ ##

Special cases

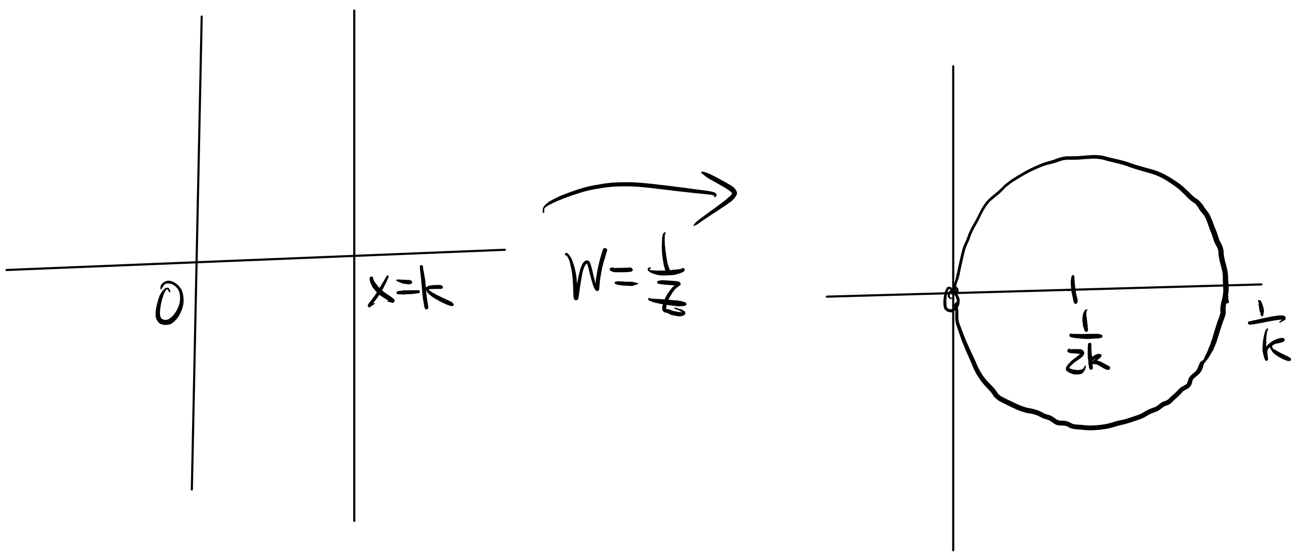
(i) Horizontal lines $y=k$ (i.e. $a=0$, $b=1$, $c=-k$)

$\Rightarrow$ circle is $s^2+\left(t+\frac{1}{2k}\right)^2=\left(\frac{1}{2k}\right)^2$

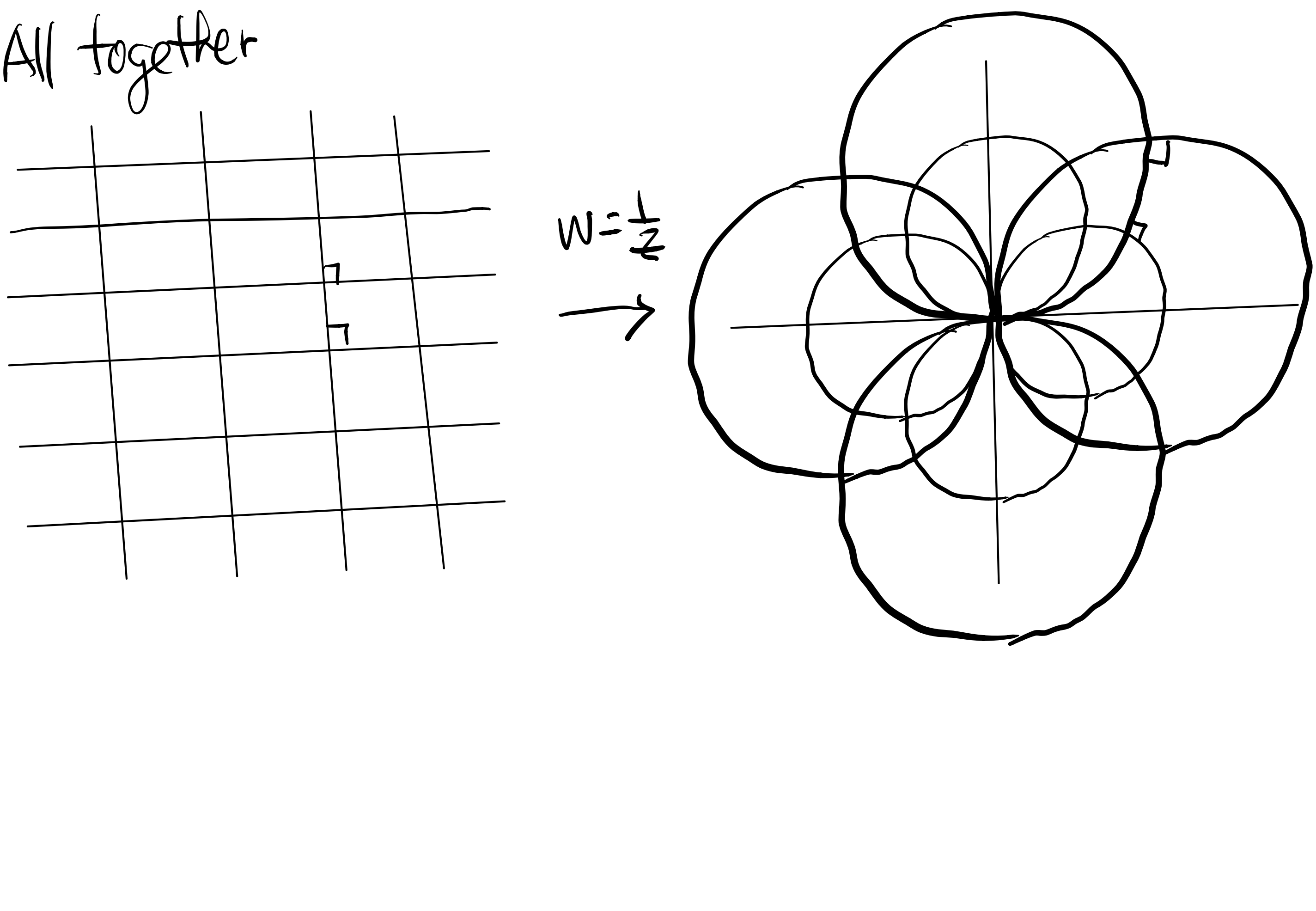


(ii) <u>Vertical lines</u>: $x=k$ (i.e. $a=1, b=0, c=-k$)

$\Rightarrow$ Circle is $\left(s-\frac{1}{2k}\right)^2+t^2=\left(\frac{1}{2k}\right)^2$



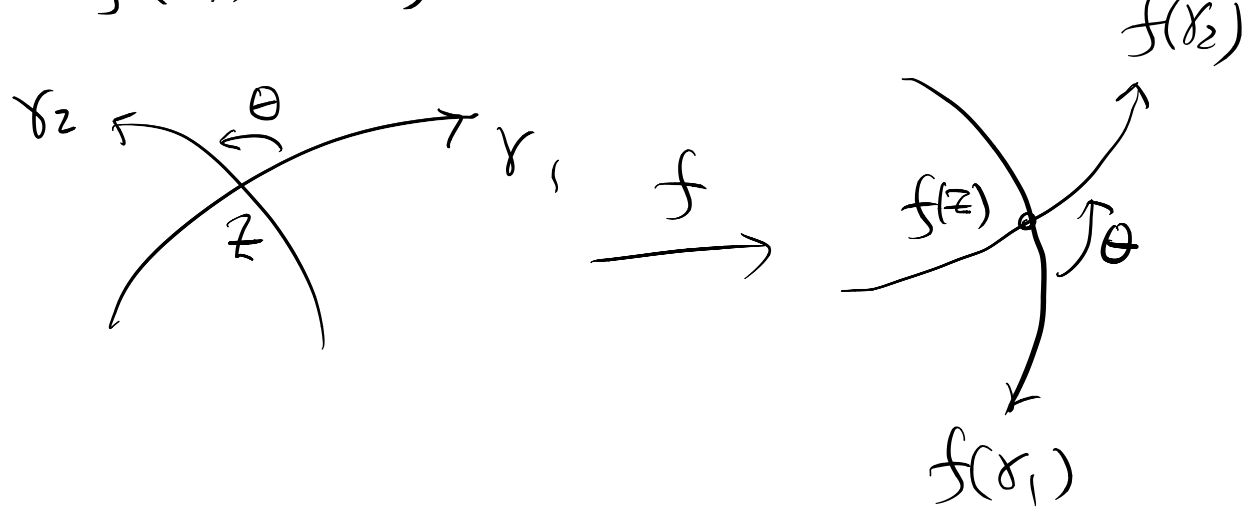
All together



Conformality (保角)

Def: A transformation f is conformal if it preserves angles.

i.e. If γ_1, γ_2 are curves passing thro. a point z, then the angle between γ_1 and γ_2 at z is the same as the angle between $f(\gamma_1)$ & $f(\gamma_1)$ at $f(z)$.



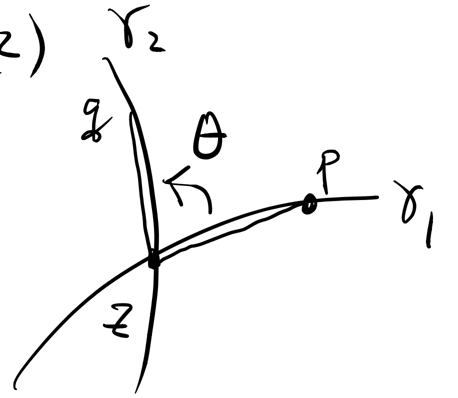
eg: Rotations, translations & homothetic transformations are conformal.

Thm: Inversion is conformal at every $z \in \mathbb{C} \setminus \{0\}$.

Pf: (1) $\frac{d}{dz} w = \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2} \ (\neq 0)$

$\Rightarrow$ conformal.

(2)



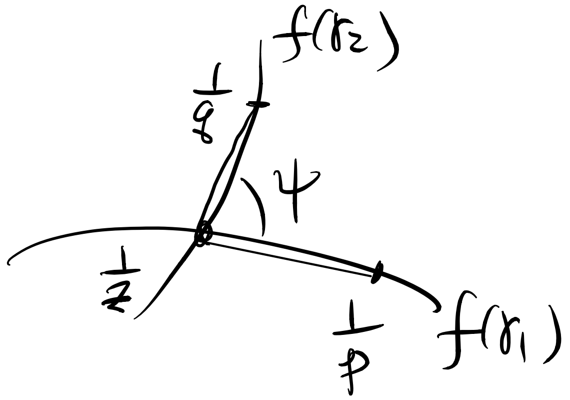
(dot product)
inner product

$$\theta = \lim_{p,q \to z} \cos^{-1}\left(\frac{p-z}{|p-z|} \cdot \frac{q-z}{|q-z|}\right)$$

$$= \lim_{p,q \to z} \cos^{-1}\left(\mathrm{Re}\left[\frac{p-z}{|p-z|}\overline{\left(\frac{q-z}{|q-z|}\right)}\right]\right)$$

$$= \lim_{p,q \to z} \cos^{-1}\left(\frac{\mathrm{Re}\left((p-z)\overline{(q-z)}\right)}{|p-z||q-z|}\right)$$

$w = f(z) = \frac{1}{z}$



Then

$$\psi = \lim_{p,q \to z} \cos^{-1}\left[\frac{\mathrm{Re}\left(\left(\frac{1}{p}-\frac{1}{z}\right)\overline{\left(\frac{1}{q}-\frac{1}{z}\right)}\right)}{\left|\frac{1}{p}-\frac{1}{z}\right|\left|\frac{1}{q}-\frac{1}{z}\right|}\right]$$

(same as $\frac{1}{p}, \frac{1}{q} \to \frac{1}{z}$)

$\therefore$

$$\psi = \lim_{p,q \to z} \cos^{-1}\left\{\frac{\mathrm{Re}\left[\frac{(z-p)\overline{(z-q)}}{pz\overline{q}\overline{z}}\right]}{\left|\frac{z-p}{pz}\right|\left|\frac{z-q}{qz}\right|}\right\}$$

$$= \lim_{p,q \to z} \cos^{-1} \left\{ \frac{\frac{1}{|z|^2} \operatorname{Re}\left[\frac{(z-p)\overline{(z-q)}}{p\overline{q}}\right]}{\frac{|z-p||z-q|}{|p||q||z|^2}} \right\}$$

$$= \lim_{p,q \to z} \cos^{-1} \left\{ \frac{\operatorname{Re}\left[\left(\frac{|p||q|}{p\overline{q}}\right)(p-z)\overline{(q-z)}\right]}{|p-z||q-z|} \right\}$$

Note that $\displaystyle \lim_{p,q \to z} \frac{|p||q|}{p\overline{q}} = \frac{|z|^2}{z\overline{z}} = 1$

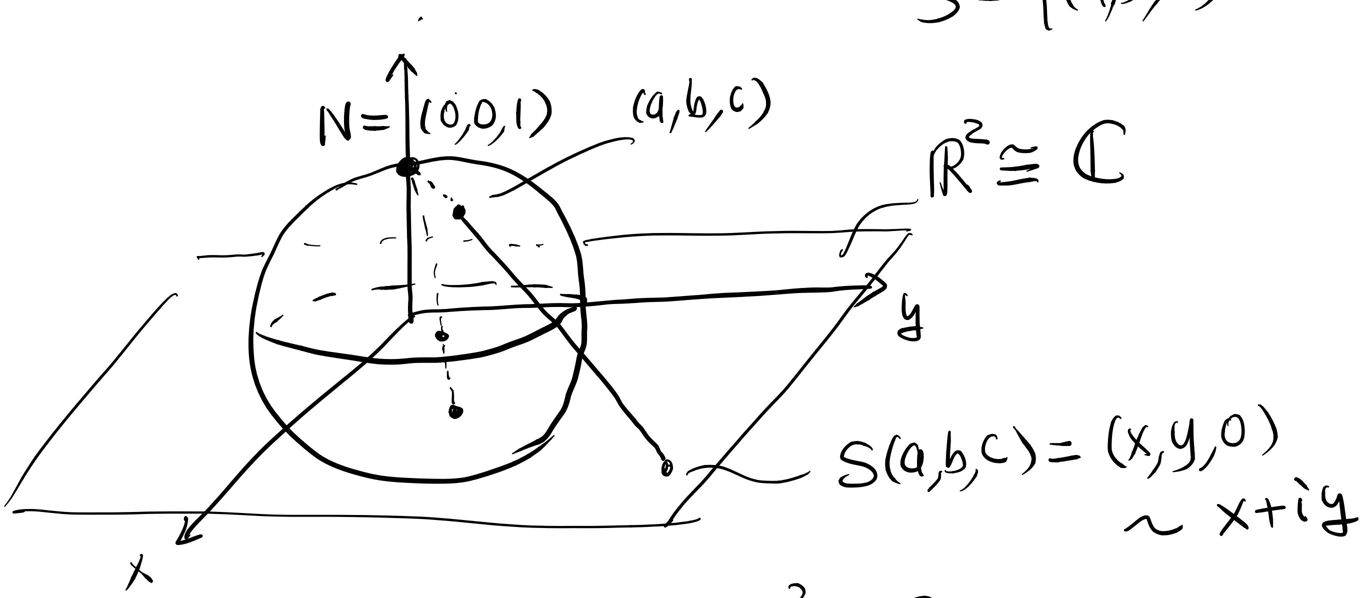
Hence

$$\psi = \lim_{p,q \to z} \cos^{-1}\left(\frac{\operatorname{Re}\left[(p-z)\overline{(q-z)}\right]}{|p-z||q-z|} \right)$$

$$= \theta . \quad \#$$

Stereographic Projection

$$\mathbb{S}^2 = \{(a,b,c) : a^2+b^2+c^2=1\}$$



$$S : \mathbb{S}^2 \setminus \{N\} \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$$

Formula for Stereographic projection:

$$\boxed{S(a,b,c) = x+iy = \frac{a+ib}{1-c}}$$

Pf: The straight line passing thro. $N=(0,0,1)$ and (a,b,c) can be parametrized by

$$(0,0,1) + t\left[(a,b,c)-(0,0,1)\right], \quad -\infty < t < +\infty.$$

For some $t=t_0$, the straight line intersects the xy-plane at $(x,y,0)$.

i.e. $(x, y, 0) = (0,0,1) + t_0 \left[(a,b,c) - (0,0,1) \right]$

$\Rightarrow \quad t_0 = \frac{1}{1-c}$ and hence

$$x = \frac{a}{1-c}, \quad y = \frac{b}{1-c} \quad ※$$

<u>Thm</u> : Stereographic projection is conformal

(Pf: Omitted)