THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5120 Topics in Geometry 2018/19
Classwork 1 Questions, solutions and remarks

Questions:

1. Let $S : \mathbb{S}^2 \backslash \{N = (0,0,1)\} \to \mathbb{C}$ be the stereographic projection given by

$$x + iy = S(a,b,c) = \frac{a+ib}{1-c}$$

Find $S^{-1} : \mathbb{C} \to \mathbb{S}^2 \backslash \{N\}$

2. Let $D = \{l : l \text{ is a straight line in } \mathbb{R}^2 = \mathbb{C}\}$ and $f : D \to \mathbb{R}$ be a function on D defined by

$$f(l) = \text{slope of } l, \text{ for } l \in D$$

   (a) Show that D is invariant in the translational geometry and in the Euclidean geometry.
   (b) Is f invariant in translational geometry? Justify your answer.
   (c) Is f invariant in Euclidean geometry? Justify your answer.

Solutions:

1. The question is asking us to find the inverse function of S, which means, expressing a, b, c in terms of x, y.

The key observation is that (a,b,c) lies on the unit sphere, so $a^2 + b^2 + c^2$ is always 1.
We are given that $x = \frac{a}{1-c}$ and $y = \frac{b}{1-c}$. Substituting $a = x(1-c)$ and $b = y(1-c)$ into $a^2 + b^2 + c^2 = 1$ to eliminate a, b, we get

$$x^2(1-c)^2 + y^2(1-c)^2 + c^2 = 1$$

Which can be re-arranged into

$$(1-c)(x^2(1-c) + y^2(1-c) - 1 - c) = 0$$

As we're excluding the north pole $N = (0,0,1)$, we have $c \neq 1$, so we get

$$x^2(1-c) + y^2(1-c) - 1 - c = 0$$

Solving for c gives $c = \frac{x^2+y^2-1}{x^2+y^2+1}$. From $a = x(1-c)$, $b = y(1-c)$, we get $a = \frac{2x}{x^2+y^2+1}$, $b = \frac{2y}{x^2+y^2+1}$.

2. (a) Since straight lines transform to straight lines under translations and rotations, the set is invariant in both translational and Euclidean geometries.
   (b) Yes. Since straight lines transform to parallel straight lines under translations, the slope is invariant in translational geometry.
   (c) No. Since slope of a straight line changes after rotation, it is not an invariant in Euclidean geometry.

Remark: Only simple geometric reasons are expected for question 2. However, most students gave computational attempts. Marks were given if the computation attempt looks like trying to prove the geometric reasons that are required to write down.

The computations were being marked. Red crosses and comments were written down when there are something wrong with the computations, but no marks were deducted for that.

For enthusiasts only: Many students attempted to computationally prove that rotating a straight line gives you a straight line, but not many succeed. For your interest, below is a computational proof that a straight line transforms into a straight line under translations and rotations. Note that you were not expected/required to do that for this classwork.

Let l be the straight line $\{z : \frac{z-z_0}{b} \in \mathbb{R}\}$, T be the transformation $z \to e^{i\theta}z + a$. Then

$$\begin{aligned} T(l) =& \{T(z) : \frac{z - z_0}{b} \in \mathbb{R}\} \\ =& \{z : \frac{T^{-1}(z) - z_0}{b} \in \mathbb{R}\} \\ =& \{z : \frac{e^{-i\theta}(z-a) - z_0}{b} \in \mathbb{R}\} \\ =& \{z : \frac{(z - (a + e^{i\theta}z_0)}{e^{i\theta}b} \in \mathbb{R}\} \end{aligned}$$

which is a straight line.