MATH1010D/1510E

Week 5 to 6 notes (preliminary version)

(Please check for any typos!)

Apart from the $+,-,\times,\div$ of derivatives, there is one more rule, which formally looks like cancellation law of fractions.

Chain Rule

If f is differentiable at $g(c)$, g is differentiable at c, then $f(g(x))$ is differentiable at c. Further, we can compute the derivative of $f(g(x))$ at c by the formula

$$\left.\frac{d\ f(g(x))}{dx}\right|_{x=c} = \left.\frac{d\ f(y)}{dy}\right|_{y=f(c)} \left.\frac{d\ y}{dx}\right|_{x=c}$$

(Here we have let $y = f(x)$).

Quick Idea on the Proof

Three steps: (i) consider the difference quotient $\frac{f(g(+h))-f(g(c))}{h} = \frac{f(g(+h))-f(g(c))}{g(c+h)-g(c)} \cdot \left(\frac{g(c+h)-g(c)}{h}\right)$, (ii) let $k = g(c+h) - g(c)$, (iii) take limit and use g is differentiable at $x = c$ implies g is continuous there.

Remarks

- Oftentimes we don't write the $|_{y=f(c)}$ or $|_{x=c}$
- Many people like to write $f(g(x))$ as $(f \circ g)(x)$.

Using Chain Rule, we can easily compute things like

Example

$$\frac{d\ e^{x^2}}{dx} = \frac{d\ e^y}{dy}\frac{dx^2}{dx} = e^y \cdot 2x = e^{x^2} \cdot 2x$$

Here we have let $y = x^2$.

We need one more tool before we can go on to describe a "simple" method to show that a certain given function is 1-1 and onto.

This tool is known as Mean Value Theorem. We introduce three of them.

The Three Mean Value Theorems

They are

(1) Rolle's Theorem, (2) Lagrange's Mean Value Theorem, (3) Cauchy's Mean Value Thereom.

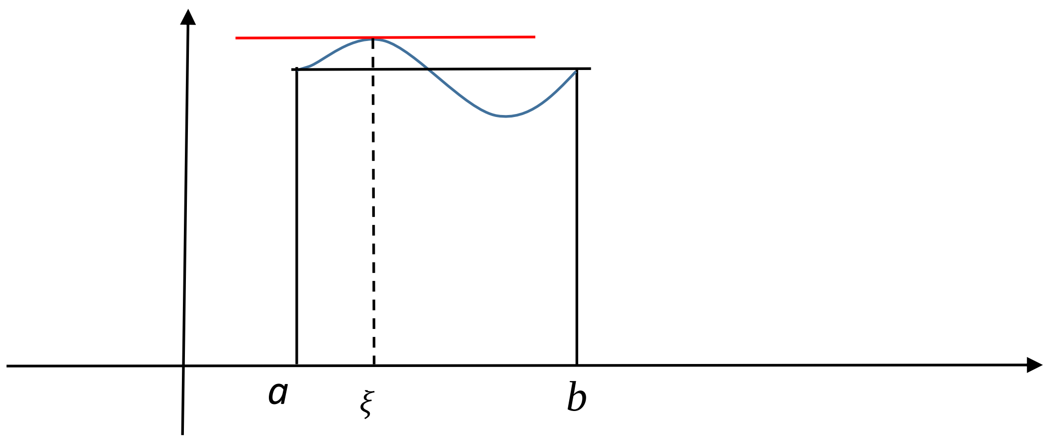
They are useful in (1) proving inequalities like $|\sin(a) - \sin(b)| \leq |a - b|$, (2) proving the L'Hôpital Rule.

Rolle's Theorem:

Assumptions

- $f(x)$ is differentiable in (a, b).
- $f(x)$ is continuous on [a,b] (This is "technical assumption", i.e. it's used to kick start the "proof")
- $f(a) = f(b)$.

Conclusion: $f'(\xi) = 0 \; \exists \xi \in (a, b)$

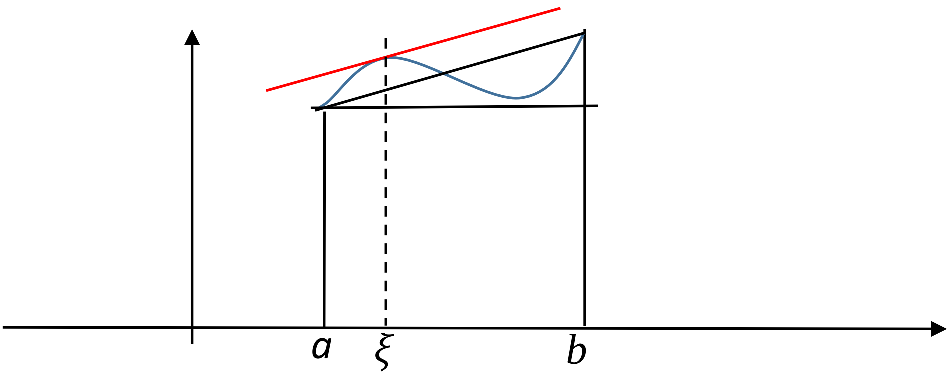


As can be seen from the picture below, Rolle's Theorem, when "rotated", gives the Lagrange's Mean Value Theorem.

Lagrange's Mean Value Theorem

It says: "If a function satisfies only (1) and (2) below, then $\exists \xi \in (a, b)$ such that:

$f'(\xi) = \frac{f(b)-f(a)}{b-a}$."



Examples for LMVT

1) Show $|\sin(a) - \sin(b)| \le |a - b|$

2) Let $a < b$, show $|\tan^{-1}(a) - \tan^{-1}(b)| \le \frac{1}{1+a^2}|a - b|$

Answers:

1) It is important to remember that we have two cases (or more?)

   Case 1: $(a \neq b)$. We can suppose that $a < b$. Consider the function $f(x) = \sin(x)$ in any domain slightly larger than the interval $[a, b]$. You can choose for example $[A, B]$ satisfying $A < a, b < B$. This will ensure that all assumptions in LMVT are satisfied.

   Case 1, Now, we use LMVT to get

   $\frac{f(b)-f(a)}{b-a} = f'(\xi)$, i.e. $\frac{\sin(b)-\sin(a)}{b-a} = \cos(\xi)\ \exists \xi \in (a, b)$.)

   Case 2: If $a = b$, then $\sin(a) - \sin(b) = 0 = b - a$, therefore the inequality is still satisfied (it is actually an "equality").

2) Consider the function $f(x) = \arctan(x)$ (in the lecture, I used the notation $\tan^{-1}(x)$, which means the same thing. I don't use this here, because it can easily lead to misunderstandings).

   Then by letting $y = \arctan(x)$, one gets $\tan(y) = x$. Now both the left-hand side and the right-hand side are functions of x, so we can differentiate both sides and get

   $\frac{d\tan(y)}{dx} = \frac{dx}{dx} = 1 \Longrightarrow \frac{d\tan(y)}{dy}\frac{dy}{dx} = 1 \Longrightarrow \sec^2(y)\ y' = 1 \Longrightarrow y' = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} =$

   $\frac{1}{1+x^2}$.

   Hence we have $\frac{d\ \arctan(x)}{dx} = \frac{1}{1+x^2}$.

   Now we apply LMVT and obtain

$$\frac{\arctan(b) - \arctan(a)}{b-a} = \frac{d\ \arctan(x)}{dx}\bigg|_{x=\xi} = \frac{1}{1+x^2}\bigg|_{x=\xi} = \frac{1}{1+\xi^2} < \frac{1}{1+a^2}$$

This is because $a < \xi$.

Conclusion: We've shown $\arctan(b) - \arctan(a) < \frac{1}{1+a^2} \times (b-a)$.

LMVT & Strictly Increasing Functions

One application of LMVT is the following result, which is useful in showing 1-1.

Theorem. Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable. Show that if $f'(x) > 0\ \forall x \in (a,b)$, then f is strictly increasing.

Proof: Pick any two numbers a, b satisfying $a < b$. Then the LMVT says that there is some $\xi \in (a,b)$ with the property that

$$\frac{f(b) - f(a)}{b-a} = f'(\xi)$$

But this means that (because $f'(\xi) > 0$ by our "positivity" assumption) the RHS is "positive". Therefore the LHS is also "positive", i.e. $\frac{f(b)-f(a)}{b-a} > 0$.

Now we know $b - a > 0$, hence it follows that $f(b) > f(a)$. That is, f is strictly increasing.

Cauchy's Mean Value Theorem

There is one more mean value theorem by the French mathematician Cauchy. This is

Cauchy's Mean Value Theorem

Assumptions:

- Let $f(x), g(x)$ be two differentiable functions in (a,b).
- Let $f(x), g(x)$ be continuous on $[a,b]$.
- Let $g'(x) \neq 0\ \ \forall x \in (a,b)$. (This guarantees that the denominator is not zero.)

Then we have the

Conclusion:

$$\exists \xi \in (a,b)\colon \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

Cauchy's MVT has many applications, one of which is L'Hôpital Rule

L'Hôpital Rule

L'Hôpital Rule says, if a limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, when $x \to c$ or $x \to \pm\infty$.

And if the limit $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Remark:

Similar conclusion holds if instead of $x \to c$, we have $x \to \infty$, or $x \to -\infty$.

Example

Find the limit $\lim_{x \to 0^+} x^x$.

Answer: The idea is to consider $e^{x \ln x}$. This leads to our studying the limit

$$\lim_{x \to 0^+} x \ln x$$

Now $x \ln x = \frac{\ln x}{\frac{1}{x}}$

So as $x \to 0^+$, the limit is of the type $\frac{0}{0}$. Therefore we are allowed to use L'Hôpital Rule.

Using L'Hôpital Rule, we get $\lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{\frac{d \ln x}{dx}}{\frac{d x^{-1}}{dx}} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} (-x) = 0^-$.

Conclusion: Putting this back into $\lim_{x \to 0^+} e^{x \ln x} = e^{\lim_{x \to 0^+} x \ln x} = e^{0^-} = 1$

Remark: You can just write 0 for the answer (of course, 0^- is more refined!).

In our previous discussion, we learned how to differentiate y with respect to x in an equation like $\tan(y) = x$.

Actually this can be done for "any" equation of the variables x, y.

There is a theorem, i.e. the Implicit Function Theorem (IFT in short), which guarantees that this can be "done" (except for some fine technical details!)
An example of how this is done is

Example

Find $y'(x)$, where x, y satisfies the following equation.

$(x^2 + y^2)^2 \;= x^2 - y^2$, then what does it mean?

Fact: IFT implies any "equation" of such type leads to $y =$ function of x (similar for $x = x(y)$).

What is really happening is that the above (equation) defines curve(s). So $y = y(x), x = x(y)$.

Answer:

We know that $(x^2 + y^2)^2 = x^2 - y^2$ implies $y = y(x)$ (meaning "y is a function of x"), hence we can differentiate both sides of the equation with respect to x and obtain

$$\frac{d\,(x^2 + y^2)^2}{dx} = \frac{d\,(x^2 - y^2)}{dx}$$

Now let $u = (x^2 + y^2)$ then the LHS (=left-hand side) becomes

$$\frac{d\,(x^2 + y^2)^2}{dx} = \frac{d\,u^2}{dx} = \frac{du^2}{du}\frac{du}{dx} = 2u \cdot \frac{d\,(x^2 + y^2)}{dx}$$

$$= 2u \cdot \left(2x + \frac{dy^2}{dx}\right) = 2(x^2 + y^2)\left(2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx}\right)$$

$$= 2(x^2 + y^2)(2x + 2y\,.\,y')$$

The RHS is equal to

$$\frac{d\,(x^2 - y^2)}{dx} = 2x - \frac{dy^2}{dx} = 2x - 2yy'$$

Putting them together, we get

$$4(x^2 + y^2)(x + 2yy') = 2x - 2yy'$$

Making y' the subject we get the answer.

Summary

What the IFT says is basically that whenever there is an equation in x, y of the form

$$F(x, y) = 0$$

then $y = y(x)$ or $x = x(y)$. (Of course, we need to make some "differentiability assumptions on F, but we will not give details here).

The ideas is consider this equation as "two equations" in 3D.

That is, $z = F(x, y)$ & $z = 0$ (In our example, $F(x, y) = (x^2 + y^2)^2 - x^2 + y^2$.)

The first equation represents a "surface" in the 3-dimensionla space, the second equation represents a horizontal plane in the 3-D space. Taken together, the two equations represent "intersection of a surface with a horizontal plane", i.e. they lead to curves.

(see the picture in the Appendix (pending))

How to show 1-1, onto for a given function

Using the "$f'(x) > 0\ \forall x \in (a,b) \Longrightarrow$ strictly increasing/decreasing", together with the following result (which we'll not prove), we can easily show that a function is 1-1, onto.

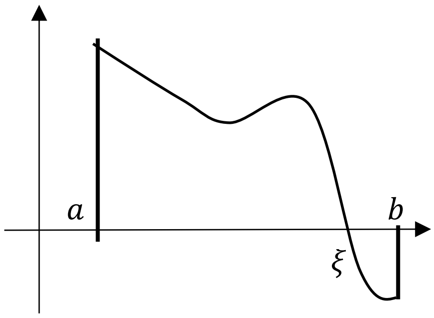
Intermediate Value Theorem

Let $f: [a,b] \to \mathbb{R}$ be continuous. Suppose also that $f(a) \cdot f(b) < 0$, then

$$f(\xi) = 0, \exists \xi \in (a,b).$$

Remark

What this theorem says is very "intuitive". It says, if f is a continuous function, whose values at $x = a$ and at $x = b$ are of "different signs" (正負號), then there must be at least one point ξ where the curve $y = f(x)$ intersects the x −axis.



Example of showing 1-1, onto

Show that the function $\tan \quad : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is 1-1, onto.

Answer: To show "onto", consider the equation $f(x) = \tan(x)$

This function is (i) continuous at every point $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$, because $\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\sin(x)$ is continuous at every such point and $\cos(x)$ also, plus $\cos(x) \neq 0$. Furthermore, the function $f(x)$ satisfies $\lim_{x \to \frac{\pi}{2}} f(x) = \infty$, $\lim_{x \to -\frac{\pi}{2}} f(x) = -\infty$, so it is onto.

Next, we show 1-1. To see this, check that $f'(x) = \sec^2(x) = \frac{1}{\cos^2(x)} > 0$

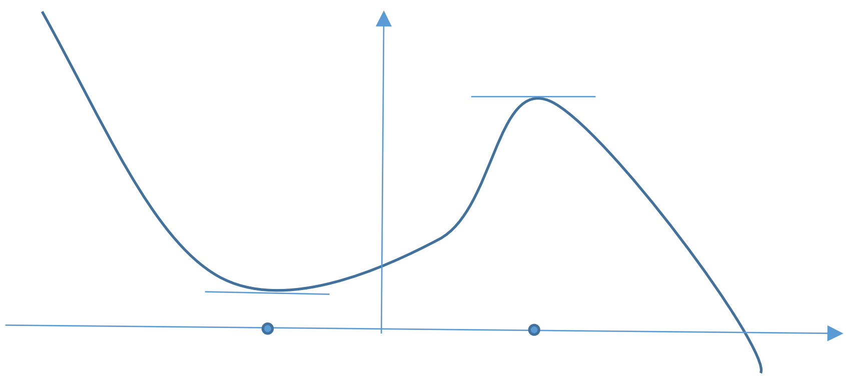
Hence the function is strictly increasing so it is 1-1.

Remark Actually the argument for onto is slightly more complicated and uses the LMVT on subintervals of the form $\left[-\frac{\pi}{2}+\frac{1}{n}, \frac{\pi}{2}-\frac{1}{n}\right]$.

Second Derivative Test – another Application of "$f'(x) > 0 \Longrightarrow$ strict increasing"

The following "second derivative test" is another application of "$f'(x) > 0 \Longrightarrow$ f strictly increasing" (similarly "$f'(x) < 0 \Longrightarrow f$ strictly decreasing")

Local Max/Min Points, Local Max/MinValues



The two blue points are local minimum/maximum points (why local? Because the function is has smallest/largest values than "nearby" points only).
At such points, say "c_1", the function has "horizontal tangent", i.e. $f'(c_1) = 0$.

At a local minimum point, to the left of it, the function is strictly decreasing, to the right strictly increasing, i.e. $f'(c_1 - sth.) < 0$,

$$f'(c_1 + sth\,) > 0$$

More precisely, $f'(c-h) < 0, f'(c+h) > 0, \forall h$ sufficiently small.
But this means the new function $f'(x)$ goes from negative to 0 to positive, i.e. the function $f'(x)$ is strictly increasing at the point "c_1".

Similar for "local maximum" point.

Summary

Let $f:(a,b)\to\mathbb{R}$ have derivative and derivative of derivative (i.e. second derivative, or $f''(x)$) at all points in (a,b), then if

1. $f'(c_1)=0$,
2. $f''(c_1)>0$

Then f has a local minimum point at c_1.

Similar for local maximum point.
Name: This is called the "second derivative" test for local maximum/minimum.
Terminologies local max/min point, local max/min value.

Example (Arithmetic Mean $\geq$ Geometic Mean)
Question: Show that $a^3+b^3+c^3\geq 3abc, a,b,c>0$ using the fact that $a^2+b^2\geq 2ab$
Answer: Consider the function $f(x)=a^3+b^3+x^3-3abx, x>0$. Our goal is to show the function is always ≥ 0.
Method:

$$f'(x)=3x^2-3ab=3(x^2-ab)$$

Putting this equal to zero, we find $f'(x)=0\Longrightarrow x=\sqrt{ab}$
Question:
Is this local max or local min point?
We check then $f''\left(\sqrt{ab}\right)=6\sqrt{ab}>0$
Hence it is local min point. Therefore, $f(x)=a^3+b^3+x^3-3abx\geq a^3+b^3$

DIY Question
Show that actually $\sqrt{ab}$ is a "global" min. point.
Answer: The function $f(x)=x^3-3abx+a^3+b^3$.
Therefore after differentiation, it becomes

$$f'(x)=3(x^2-ab)=3(x-(-\sqrt{ab}\,))(x-\sqrt{ab}\,)$$

This implies that $f'(x)>0$, if $x>\sqrt{ab}$, i.e. f is strictly increasing if $x>\sqrt{ab}$.
Similarly, one sees that f is strictly decreasing if $-\sqrt{ab}<x<\sqrt{ab}$.
In our case, $0<x,$ so it remains to check the "strict increasing/decreasingness" of f for x satisfying $0<x<\sqrt{ab}$. But in this region, f is "strictly decreasing".

Conclusion: Since f is strictly decreasing for $0<x<\sqrt{ab}$ and strictly increasing for $\sqrt{ab}<x$, therefore $\sqrt{ab}$ is an absolute (some people call it "global") min. point.

Another Way of understanding LMVT = Taylor's Theorem

LMVT says $\frac{f(b)-f(a)}{b-a}=f'(\xi)$ for functions satisfying certain conditions. Now let's make the following changes:

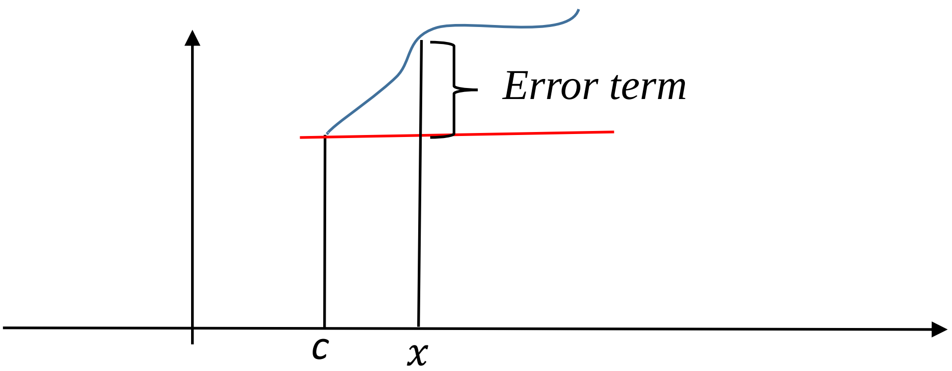
- Change b to x,
- Change a to c.

Then we obtain $\frac{f(x)-f(c)}{x-c} = f'(\xi)\ \exists\xi \in (c,x)$ or (x,c)

Rearranging the terms, we obtain $f(x) = f(c) + f'(\xi)(x-c) = f(c) + \frac{f'(\xi)}{1!}(x-c)$,

which means LHS (i.e. $y = f(x)$) is equal to $y = f(c)$ plus an error term of the form $\frac{f'(\xi)}{1!}(x-c)^1$.

Picture



Remarks

- The blue curve is the curve given by $y = f(x)$.
- The red line is the line given by $y = f(c)$.
- The error term is the term given by $\frac{f'(\xi)}{1!} \cdot (x-c)$.
- The error term becomes 0, if $x = c$.
- The point c is called the "center".
- This approximation of the curve $y = f(x)$ by the line $y = f(c)$ is too crude. Taylor' Theorem says that we can do better and have the formula

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \text{error term}$$

Here the error term is given by $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$.