Q.1

If $f \in \mathcal{R}[a,b]$ and $(\dot{P}_n)$ is any sequence of tagged partitions of $[a,b]$ such that $\|\dot{P}_n\| \to 0$, prove that $\int_a^b f = \lim_n S(f;\dot{P}_n)$.

Solution:

Let $\varepsilon > 0$ be fixed & δ_ε be the corresponding constant specified in the definition 7.1.1 (i.e. Riemann integrability)

Since $\|\dot{P}_n\| \to 0$, $\exists N$ s.t. $\|\dot{P}_n\| < \delta_\varepsilon$ as $n \geq N$.

Then since $f \in \mathcal{R}[a,b]$, by def. of Riemann integrability,

$$\left| S(f;\dot{P}_n) - \int_a^b f \right| < \varepsilon$$

as $n \geq N$

$\therefore \lim_{n\to\infty} S(f;\dot{P}_n) = \int_a^b f$ by def. of limit.

Q.2

Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := \frac{1}{x}$ if $x \in [0,1]$ is irrational. Explain why $g \notin R[0,1]$. However, show that there exists a sequence $(\dot{P}_n)$ of tagged partitions of $[a,b]$ such that $\|\dot{P}_n\| \to 0$ and $\lim_n S(g; \dot{P}_n)$ exists.

Solution:

Consider $\dot{P}_n^1 = \{([\frac{i-1}{n}, \frac{i}{n}], r_n^i)\}_{i=1}^n$

$\dot{P}_n^2 = \{([\frac{i-1}{n}, \frac{i}{n}], s_n^i)\}_{i=1}^n$

where $r_n^i \in [\frac{i-1}{n}, \frac{i}{n}] \cap \mathbb{Q}$

$s_n^i \in [\frac{i-1}{n}, \frac{i}{n}] \cap \mathbb{Q}^c$

thm 2.4.8 & Cor 2.4.9

(which exist because of the density of $\mathbb{Q}$ & $\mathbb{Q}^c$ in $\mathbb{R}$)

Then $\|\dot{P}_n^1\|, \|\dot{P}_n^2\| \to 0$ as $n \to \infty$.

Suppose in contrast that $g \in R[0,1]$. Then by Q.1,

$\lim_{n\to\infty} S(g, \dot{P}_n^1) = \lim_{n\to\infty} S(g, \dot{P}_n^2)$.

However, $S(g, \dot{P}_n^1) = \sum_{i=1}^n g(r_n^i) \cdot \frac{1}{n} = 0$ (This also show the 2nd part)

$S(g, \dot{P}_n^2) = \sum_{i=1}^n g(s_n^i) \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^n \frac{1}{s_n^i} \geq \frac{1}{n} \sum_{i=1}^n \frac{n}{i} = \sum_{i=1}^n \frac{1}{i}$

$\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{i} = \infty$ (refer to example 3.7.6)

$\therefore \lim_{n\to\infty} S(g, P_n^2) = \infty$

$\therefore$ Contradiction to $g \in R[0,1]$.

$\therefore g \notin R[0,1]$.

Q.3

If $f \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-translate of f.

Solution:

Let $\varepsilon > 0$ be fixed & δ_ε be the corresponding constant in the def. of Riemann integrability of f.

Now, consider a tagged partition $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ of $[a+c, b+c]$ with $\|\dot{P}\| < \delta_\varepsilon$. Since length of an interval is translation invariant, $\dot{P}' = \{([x_{i-1}-c, x_i - c], t_i - c)\}_{i=1}^{n}$ is a tagged partition of $[a,b]$ with $\|\dot{P}'\| < \delta_\varepsilon$.

By def. of Riemann integrability of f, we have

$$\left| S(f, \dot{P}') - \int_a^b f \right| < \varepsilon.$$

Note that

$$S(f, \dot{P}') = \sum_{i=1}^{n} f(t_i - c)\,(x_i - c - (x_{i-1} - c))$$

$$= \sum_{i=1}^{n} g(t_i)\,(x_i - x_{i-1})$$

$$= S(g, \dot{P})$$

$\therefore \quad |S(g, \dot{P}) - \int_a^b f| < \varepsilon.$

$\therefore \quad g \in R[a+c, b+c]$ & $\int_{a+c}^{b+c} g = \int_a^b f$ by def. of Riemann integrability.