Q.1

Suppose that f and g are continuous on $[a,b]$, differentiable on (a,b), that $c \in [a,b]$ and that $g(x) \neq 0$ for $x \in [a,b]$, $x \neq c$. Let $A = \lim_{x \to c} f$ and $B = \lim_{x \to c} g$. If $B = 0$, and if $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists in $\mathbb{R}$, show that we must have $A = 0$.

Solution:

Note that for $x \in (a,b)$, $x \neq c$, $f(x) = \frac{f(x)}{g(x)} \cdot g(x)$.

$\lim_{x \to c} g(x)$ exists & is equal to $g(c)$ since g is continuous at c.

By assumption, $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists.

Therefore, $A = \lim_{x \to c} f(x)$

$$= \lim_{x \to c} \left(\frac{f(x)}{g(x)} \cdot g(x) \right)$$

$$= \left(\lim_{x \to c} \frac{f(x)}{g(x)} \right) \left(\lim_{x \to c} g(x) \right)$$

$= 0$ since $B = \lim_{x \to c} g(x) = 0$

Q.2

In addition to the suppositions of the preceding exercise, let $g(x) > 0$ for $x \in [a, b]$, $x \neq c$. If $A > 0$ and $B = 0$, prove that we must have $\lim_{x \to c} \frac{f(x)}{g(x)} = \infty$. If $A < 0$ and $B = 0$, prove that we must have $\lim_{x \to c} \frac{f(x)}{g(x)} = -\infty$.

Solution:

By def. of limit,

$\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $|f(x) - A| < \varepsilon_1$ as $0 < |x - c| < \delta_1$ — ①

$\forall \varepsilon_2 > 0, \exists \delta_2 > 0$ s.t. $|g(x) - 0| < \varepsilon_2$ as $0 < |x - c| < \delta_2$ — ②

$\boxed{\text{when } A > 0 \ \& \ B = 0:}$

Choose $\varepsilon_1 = \frac{A}{2} > 0$.

By ①, when $0 < |x - c| < \delta_1$, (δ_1 is fixed for the choice $\varepsilon_1 = \frac{A}{2}$)

$-\frac{A}{2} < f(x) - A < \frac{A}{2}$

$\Rightarrow f(x) > \frac{A}{2}$ ③

Moreover, by ② & assumption, $0 < g(x) < \varepsilon_2$ as $0 < |x - c| < \delta_2$. ④

Given $M > 0$, we can choose $0 < \varepsilon_2 < \frac{A}{2M}$, so that $\frac{A}{2\varepsilon_2} > M$.

For this particular ε_2, we can fix a δ_2 so that ④ holds.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x - c| < \delta$, then both ③ & ④ holds.

Then $\frac{f(x)}{g(x)} > \frac{A/2}{\varepsilon_2} > M$. Hence, $\lim_{x \to c} \frac{f(x)}{g(x)} = +\infty$ by def.

$\boxed{\text{When } A<0 \ \& \ B=0:}$

Choose $\varepsilon_1 = -\frac{A}{2} > 0$.

By ①, when $0<|x-c|<\delta_1$, (δ_1 is fixed for the choice $\varepsilon = -\frac{A}{2}$)

$$\frac{A}{2} < f(x) - A < -\frac{A}{2}$$

$$\Rightarrow f(x) < \frac{A}{2} \qquad ③$$

Moreover, by ② & assumption, $0<g(x)<\varepsilon_2$ as $0<|x-c|<\delta_2$. ④

Given $M<0$, we can choose $0<\varepsilon_2<\frac{A}{2M}$, so that $\frac{A}{2\varepsilon_2}<M$.

For this particular ε_2, we can fix a δ_2 so that ④ holds.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $0<|x-c|<\delta$, then both ③ & ④ holds.

Then $\frac{f(x)}{g(x)} < \frac{A/2}{\varepsilon_2} < M$. Hence, $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = -\infty$ by def.

Q.3

Try to use L'Hôpital's Rule to find the limit of $\frac{\tan x}{\sec x}$ as $x \to \frac{\pi}{2}^-$. Then evaluate directly by changing to sines and cosines.

Solution:

Consider $f(x) = \tan x$, $x \in (0, \frac{\pi}{2})$, which are differentiable.

$g(x) = \sec x$, $x \in (0, \frac{\pi}{2})$

$\lim\limits_{x \to \frac{\pi}{2}^-} \sec x = +\infty$, $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x = +\infty$

Let $L = \lim\limits_{x \to \frac{\pi}{2}^-} \frac{f(x)}{g(x)}$.

$$L = \lim_{x \to \frac{\pi}{2}^-} \frac{f'(x)}{g'(x)} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan x} = \frac{1}{L}$$

$\therefore L^2 = 1 \Rightarrow L = 1$ since $\sec x, \tan x > 0$ on $(0, \frac{\pi}{2})$

Direct calculation:

$$\lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{\sec x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} \cdot \cos x = \lim_{x \to \frac{\pi}{2}^-} \sin x = \sin \frac{\pi}{2} = 1.$$