1. Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

Solution:

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^{1/3} - 0}{x - 0} = x^{-2/3}$$

Claim: $\lim_{x \to 0} x^{-2/3}$ does not exist.

Proof: Suppose that $\lim_{x \to 0} x^{-2/3}$ exists and equal to $M \in \mathbb{R}$.

Then $\lim_{x \to 0^+} x^{-2/3} = M$.

By def., $\forall \varepsilon > 0, \exists \delta > 0$ s.t. when $0 < x < \delta$, $|x^{-2/3} - M| < \varepsilon$. (*)

$\forall L > 0$, $x^{-2/3} > L$ when $0 < x < L^{-3/2}$.

If $L > M+2$, $x^{-2/3} > L > M+2$.

$\therefore$ $|x^{-2/3} - M| = x^{-2/3} - M > 2$ when $0 < x < L^{-3/2}$.

$\Rightarrow \forall \delta > 0$, $|x^{-2/3} - M| > 2$ for some $0 < x < \delta$

(By choosing $0 < x < \min\{\delta, L^{-3/2}\}$.)

$\Rightarrow$ contradiction to (*). $\square$

2. Let $n \in \mathbb{N}$ and let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x^n$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$. For which values of n is f' continuous at 0? For which values of n is f' differentiable at 0?

Solution:

At each $x = a > 0$, $f(x) = x^n \; \forall x \in (a - \frac{a}{2}, a + \frac{a}{2})$

$\therefore f'(x) = n x^{n-1} \; \forall x > 0$

Similarly, $f'(x) = 0 \; \forall x < 0$.

When $n = 1$, $f'(x) = 1$. $\forall x > 0$

Since $\lim_{x \to 0^+} f'(x) = 1 \neq 0 = \lim_{x \to 0^-} f'(x)$, $f'(x)$ is not continuous.

When $n = 2$, $f'(x) = 2x \; \forall x > 0$.

$$\left.\begin{aligned} \lim_{x \to 0^+} f'(x) &= \lim_{x \to 0^+} 2x = 0 \\ \lim_{x \to 0^-} f'(x) &= \lim_{x \to 0^-} 0 = 0 \end{aligned}\right\} \Rightarrow \lim_{x \to 0} f'(x) = 0$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 - 0}{x - 0} = \lim_{x \to 0^+} x = 0$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{0 - 0}{x - 0} = 0$$

$\therefore f'(0) = 0 = \lim_{x \to 0} f'(x) = 0$

$\therefore f'$ is cont. at $x = 0$.

$$\lim_{x\to 0^+} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0^+} \frac{2x-0}{x-0} = \lim_{x\to 0^+} 2 = 2$$

$$\lim_{x\to 0^-} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0 \neq 2$$

$\therefore$ f' is not differentiable at 0.

When $n \geqslant 3$,

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} nx^{n-1} = 0$$

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} 0 = 0$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x^n-0}{x-0} = \lim_{x\to 0^+} x^{n-1} = 0$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0-0}{x-0} = 0$$

$\therefore$ $f'(0) = \lim_{x\to 0} f'(x)$.

$\therefore$ f' is continuous at 0.

$$\lim_{x\to 0^+} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0^+} \frac{nx^{n-1}-0}{x-0} = \lim_{x\to 0^+} nx^{n-2} = 0$$

$$\lim_{x\to 0^-} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0$$

$\therefore$ f' is diff. at 0.

3. Determine where $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + |x|$ is differentiable and find the derivative.

Solution:

Rewrite
$$g(x) = \begin{cases} 2x + x = 3x, & x \geqslant 0 \\ 2x - x = x, & x < 0 \end{cases}$$

g is differentiable on $x > 0$ since $g(x) = 3x \ \forall x > 0$.
$$\Rightarrow g'(x) = 3 \ \forall x > 0.$$

g is differentiable on $x < 0$ since $g(x) = x \ \forall x < 0$
$$\Rightarrow g'(x) = 1 \ \forall x < 0$$

It is left to check whether g is differentiable at 0.

$$\lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^+} \frac{3x - 0}{x - 0} = 3$$

$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^-} \frac{x - 0}{x - 0} = 1 \neq 3$$

$\therefore g$ is not differentiable at 0.

4. If $f: \mathbb{R} \to \mathbb{R}$ is diff. at $c \in \mathbb{R}$, show that

$$f'(c) = \lim_{n\to\infty} n(f(c+\tfrac{1}{n}) - f(c))$$

Give an counter-example for a function whose limit above exists, but f is not differentiable at c.

Solution:

$f'(c) = \lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$ exists since f is diff. at c.

Consider the sequence $(\frac{1}{n})$.

Note that $\lim_{n\to\infty} \frac{1}{n} = 0$ & $\frac{1}{n} \neq 0 \ \forall n \in \mathbb{N}$.

By the sequential criterion,

$$f'(c) = \lim_{n\to\infty} \frac{f(c+\frac{1}{n}) - f(c)}{1/n}$$

$$= \lim_{n\to\infty} n\{f(c+\tfrac{1}{n}) - f(c)\}$$

Counter-example:

$f(x) = |x|$

f is not diff. at $x=0$. (Exercise)

$$\lim_{n\to\infty} n\{f(\tfrac{1}{n}) - f(0)\} = \lim_{n\to\infty} n\{|\tfrac{1}{n}| - 0\}$$

$$= \lim_{n\to\infty} n \cdot \frac{1}{n}$$

$$= 1$$

If $f: \mathbb{R} \to \mathbb{R}$ is diff. at $c \in \mathbb{R}$, show that

$$f'(c) = \lim_{n\to\infty} \left(n \left\{ f(c+\tfrac{1}{n}) - f(c) \right\} \right)$$

Counter-example for existence of limit $\not\Rightarrow$ diff.

$f(x) = |x|$

$n\left(|\tfrac{1}{n}| - 0\right)$