MATH2060 Solution 9

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8.3 Q3

The inequality trivially holds When $x = 0$. Now assume $x > 0$. By Taylor theorem for e^x on $[0, x]$, there exists some $x_0 \in (0, x)$ such that

$$e^x = \sum_{k=0}^{n-1} \frac{e^{(k)}(0)}{k!}(x-0)^k + \frac{e^{(n)}(x_0}{n!}(x-0)^n = 1 + \frac{x}{1!} + ... + \frac{x^{n-1}}{(n-1)!} + \frac{e^{x_0}x^n}{n!}$$

Since e^x is increasing and $0 \leq x_0 \leq a$, we have $1 \leq e^{x_0} \leq e^a$. Then the inequality follows.

8.3 Q8

First we suppose $f(0) \neq 0$. Consider the function $g(x) = \frac{f(x)}{f(0)}$. The function g satisfies $g(0) = 1$ and $g'(x) = g(x)$ for any $x \in \mathbf{R}$. By the uniqueness result of such function (Theorem 8.3.4), $g(x) = e^x$ for any $x \in \mathbf{R}$. If $f(0) = 0$, the same proof in Theorem 8.3.4 applies and we can conclude that $f \equiv 0$.

8.3 Q9

First note that

$$\sum_k x_k = (\frac{1}{A}\sum_k a_k) - n = 0.$$

Using the inequality $1 + x \leq e^x$, we have $0 < \frac{a^k}{A} = 1 + x_k \leq e^{x_k}$ for any k. Multiplying these terms, we have

$$a_1 \ldots a_n \leq A^n e^{\sum_k x_k} = A^n.$$

The equality holds when $1 + x_k = e^{x_k}$ for each k. This is equivalent to say $x_k = 0$ for any k. Hence the equality holds if and only if $a_1 = \cdots = a_n$.

8.4 Q2

This is a direct consequence of Pythagorean identity (8.4.3.).

8.4 Q3

Suppose $x \le 0$. By Theorem 8.4.8, we have $-(-x) \le \sin(-x) \le -x$ as $-x \ge 0$. Since $\sin(-x) = -\sin x$, we have $x \le \sin x \le -x$ for $x \le 0$. This proves $|\sin x| \le |x|$ for $x \in \mathbf{R}$. Similarly, by Theorem 8.4.8, we have $-\frac{x^3}{6} \le \sin(-x) - (-x) \le 0$ for $x \le 0$. It follows that $|\sin x - x| \le \frac{|x|^3}{6}$ for all $x \in \mathbf{R}$.

8.4 Q4

By Theorem 8.4.8, we have $\cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ for $x \ge 0$. Then we have

$$\sin x = \int_0^x \cos t dt \le \int_0^x 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 dt = x - \frac{1}{6}x^3 + \frac{1}{120}x^5.$$

Moreover, we have

$$\cos x - 1 = -\int_0^x \sin t dt \ge -\int_0^x t - \frac{1}{6}t^3 + \frac{1}{120}t^5 dt = -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6.$$

Set the polynomial $P(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$. Note that for $0 \le x \le 2$,

$$P'(x) = -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 \le x(-1 + \frac{1}{6}x^2) \le 0.$$

Hence $P(x)$ is decreasing on $[0, 2]$. Try to find the first positive root of P. Note that $P(1.5) \approx 0.07 > 0$. Hence we know $\cos x \ge P(x) > 0$ for $x \in [0, 1.5]$.

As $\frac{\pi}{2}$ is the smallest positive root of $\cos x$, we can conclude that $\pi \ge 3$. one can get more precise estimates by substantial calculation.