MATH2060 Solution 5

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7.2 Q1

It follows directly from Cauchy Criterion 7.2.1. (Let $\eta = \frac{1}{n}$ for $n \in \mathbb{N}$.)

7.2 Q3

Define the set $A = \{\frac{1}{k} : k \in \mathbb{N}\}$ and let $n \in \mathbb{N}$. Suppose $\dot{\mathcal{P}}_n = \{([0, \frac{1}{n+1}], \frac{1}{n+1})\} \cup \{([\frac{i}{n+1}, \frac{i+1}{n+1}], t_i)\}_{i=1}^n$ with $t_i \notin A$. Suppose $\dot{\mathcal{Q}}_n = \{([\frac{i}{n+1}, \frac{i+1}{n+1}], t_i)\}_{i=0}^n$ with $t_i \notin A$. Note that $||\dot{\mathcal{P}}_n|| = ||\dot{\mathcal{Q}}_n|| = \frac{1}{n+1} < \frac{1}{n}$. Also, we have

$$S(H; \dot{\mathcal{P}}_n) = H(\frac{1}{n+1})(\frac{1}{n+1} - 0) + \sum_{i=1}^{n} H(t_i)(\frac{i+1}{n+1} - \frac{i}{n+1}) = \frac{n+1}{n+1} + 0 = 1.$$

$$S(H; \dot{\mathcal{Q}}_n) = \sum_{i=0}^{n} H(t_i)(\frac{i+1}{n+1} - \frac{i}{n+1}) = 0.$$

Let $\epsilon_0 = \frac{1}{2}$. Then we have $|S(H; \dot{\mathcal{P}}_n) - S(H; \dot{\mathcal{Q}}_n)| = 1 > \epsilon_0$. By Q1, H is not integrable.

7.2 Q6

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

For any internal I, there exist $x_1 \in I \cap \mathbb{Q}$ and $x_2 \in I \setminus \mathbb{Q}$. Hence f is not constant in I and thus f cannot be a step function.

7.2 Q7

Let $f : [a, b] \to \mathbb{R}$ and $S(f; \dot{\mathcal{P}})$ be a Riemann sum of f with the tagged partition $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$. Then we can define the function $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = \begin{cases} f(t_i) & x \in [x_{i-1}, x_i), i < n \\ f(t_n) & x \in [x_{n-1}, x_n] \end{cases}$$

Then ϕ is a step function with

$$\int_a^b \phi = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = S(f; \dot{\mathcal{P}}).$$

7.2 Q13

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x > 0 \end{cases}$$

For any $c \in (0, 1)$, since f is continuous on $[c, 1]$, we have $f \in \mathcal{R}[c, 1]$. However, since f is not bounded on $[0, 1]$, it follows that $f \notin \mathcal{R}[0, 1]$.

7.2 Q16

Denote the maximum of f on $[a, b]$ by M and the minimum of f on $[a, b]$ by m. Then we have

$$m(b - a) = \int_a^b m \leq \int_a^b f \leq \int_a^b M = M(b - a).$$

Since f is continuous on $[a, b]$, and $m \leq \frac{\int_a^b f}{b-a} \leq M$, by Intermediate Value Theorem, there exists some $c \in [a, b]$ such that $f(c) = \frac{\int_a^b f}{b-a}$.