MATH2060 Solution

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6.2 Q5

Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = x^{1/n} - (x-1)^{1/n}$. Note that

$$f'(x) = \frac{1}{n}(x^{\frac{1-n}{n}} - (x-1)^{\frac{1-n}{n}}).$$

Since $n \geq 2$, we have $\frac{1-n}{n} < 0$. Then for all $x > 1$, with the fact that $x > x-1 > 0$, we have $x^{\frac{1-n}{n}} - (x-1)^{\frac{1-n}{n}} < 0$. This proves that $f'(x) < 0$ for $x > 1$. By Mean Value Theorem, there exists some $c \in (1, a/b)$ such that

$$f(a/b) - f(1) = f'(c)(a/b - 1) < 0.$$

Rearranging the terms, we get

$$a^{1/n} - b^{1/n} < (a-b)^{1/n}.$$

6.2 Q7

Let $f(x) = \ln x$. For any $x > 1$, by Mean Value Theorem for $f(x)$ on $[1, x]$, there exists some $c \in (1, x)$ such that

$$\ln x = f(x) - f(1) = f'(c)(x-1) = \frac{1}{c}(x-1).$$

Since $1 < c < x$, we know $\frac{1}{x} < \frac{1}{c} < 1$. Hence for any $x > 1$, we have

$$\frac{x-1}{x} < \ln x < x - 1.$$

6.2 Q20

(a). Apply Mean Value Theorem to f on $[0, 1]$ and we have

$$f(1) - f(0) = f'(c_1)(1-0)$$

for some $c_1 \in (0, 1)$. Since $f(0) = 0$ and $f(1) = 1$, we know $f'(c_1) = 1$.

(b). Apply Mean Value Theorem to f on $[1,2]$ and we have

$$f(2) - f(1) = f'(c_2)(2-1)$$

for some $c_2 \in (1,2)$. Since $f(1) = 1$ and $f(2) = 1$, we know $f'(c_2) = 0$.

(c). Since $0 < c_1 < 1 < c_2 < 2$, we know $f(x)$ is differentiable on $[c_1, c_2]$. By Darbous's Theorem, there exists some $c \in (c_1, c_2) \subset (0,2)$ such that $f'(c) = \frac{1}{3}$ as $f'(c_2) < \frac{1}{3} < f'(c_1)$.

6.3 Q4

Note that $f(0) = g(0) = 0$ and $g'(0) = 1 \neq 0$. Next we show $f(x)$ is differentiable at $x = 0$. Let $\epsilon > 0$ and choose $\delta = \epsilon$. If $|x| < \delta$, we have

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \leq \left| \frac{x^2}{x} \right| = |x| \leq \epsilon.$$

Hence $f'(0) = 0$. We can now apply Theorem 6.3.1 (twice from both sides) to get

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = 0.$$

One cannot apply Theorem 6.3.3 here simply because there exists no neighborhood of 0 where $f(x)$ is differentiable.

6.3 Q5

Since $-1 \leq \sin(1/x) \leq 1$, by Squeeze Theorem, we have

$$\lim_{x \to 0} x \sin(1/x) = 0.$$

Then we can conclude that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x} = \lim_{x \to 0} \frac{x \sin(1/x)}{\frac{\sin x}{x}} = \frac{0}{1} = 0.$$

If $x \neq 0$, then $g'(x) = \cos x$ and $f'(x) = 2x \sin(1/x) - \cos(1/x)$. Choose $a_n = 1/(2\pi n)$ and $b_n = 1/(\pi + 2\pi n)$. Then $\frac{f'(a_n)}{g'(a_n)} = -1/\cos a_n$ and $\frac{f'(b_n)}{g'(b_n)} = 1/\cos b_n$. By continuity of cosine function, we have

$$\lim_{n \to \infty} \frac{f'(a_n)}{g'(a_n)} = \lim_{x \to \infty} \frac{-1}{\cos a_n} = -1. \quad \lim_{n \to \infty} \frac{f'(b_n)}{g'(b_n)} = \lim_{x \to \infty} \frac{1}{\cos b_n} = 1.$$

By sequential criterion, we know $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ does not exist.

6.3 Q12

By Theorem 6.3.5 (L'Hospital's Rule II), we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x f(x)}{e^x} = \lim_{x\to\infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x\to\infty} f(x) + f'(x) = L.$$

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}(f(x) + f'(x)) - \lim_{x\to\infty} f(x) = L - L = 0.$$