Power Series

Def 9.4.7 If $f_n(x) = a_n(x-c)^n$, a_n & $c \in \mathbb{R}$, $\forall n = 0, 1, 2, \cdots$

then
$$\sum f_n(x) = \sum a_n (x-c)^n$$
is called a <u>power series around $x=c$</u>.

<u>Remarks</u>:
- Power series usually starts with $n=0$ (instead of $n=1$):
$$\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
- $\sum a_n x^n$ may <u>not</u> defined over all of $\mathbb{R}$:

(i) $\sum_{n=0}^{\infty} n!\, x^n$ converges only for $x=0$, (Ex!)

(ii) $\sum_{n=0}^{\infty} x^n$ converges for $|x|<1$, (geometric series)

(iii) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges $\forall x \in \mathbb{R}$, (exponential function)

Hence there is a need to determine the <u>set</u> on which $\sum a_n x^n$ converges.

In the following, we consider the case that "$c=0$".
This is no loss of generality as the "translation $y = x-c$" reduces $\sum a_n (x-c)^n$ to $\sum a_n y^n$.

<u>Recall</u>: (Def 3.4.10 & Thm 3.4.11)

For (x_n) a <u>bounded</u> seq., <u>limit superior</u> of (x_n):

$$\limsup x_n \overset{\text{def}}{=} \inf\{v \in \mathbb{R} : v < x_n \text{ for finitely many } n\}$$
$$= \inf\{v \in \mathbb{R} : x_n \leq v \text{ for sufficiently large } n\}$$

And (i) If $v > \limsup x_n$, then

$x_n \leq v$ for sufficiently large n,

i.e. $\exists\, K(v) \in \mathbb{N}$ s.t. if $n \geq K(v)$, then $x_n \leq v$.

(ii) If $w < \limsup x_n$, then $\exists$ <u>infinitely</u> many $n \in \mathbb{N}$

s.t. $w \leq x_n$.

<u>Def 9.4.8</u> Let
- $\sum a_n x^n$ be a power series, and
- $\rho = \begin{cases} \limsup(|a_n|^{\frac{1}{n}}), & \text{if } (|a_n|^{\frac{1}{n}}) \text{ is a bdd seq.} \\ +\infty, & \text{otherwise} \end{cases}$

Then
- the <u>radius of convergence</u> of $\sum a_n x^n$ is defined by

$$R = \frac{1}{\rho} = \begin{cases} 0, & \text{if } \rho = +\infty \\ \dfrac{1}{\limsup |a_n|^{\frac{1}{n}}}, & \text{otherwise} \left(\begin{array}{l}\text{including } R = +\infty \\ \text{when } \limsup |a_n|^{\frac{1}{n}} = 0\end{array}\right) \end{cases}$$

- The <u>interval of convergence</u> is the <u>open</u> interval $(-R, R)$

Thm 9.4.9 (Cauchy-Hadamard Theorem)

If R is the radius of convergence of $\sum a_n x^n$, then

$\sum a_n x^n$ is $\begin{cases} \bullet \text{ absolutely convergent if } |x|<R, \\ \bullet \text{ divergent if } |x|>R. \end{cases}$

Remark: No conclusion for $|x|=R$:

(i) $\sum x^n$: $\rho = \limsup |a_n|^{\frac{1}{n}} = \limsup 1 = 1$

$\Rightarrow R = \frac{1}{\rho} = 1.$

$\begin{cases} x=1: & \sum x^n = 1+1+1+\cdots \text{ is divergent} \\ x=-1: & \sum x^n = -1+1-1+1 \text{ is divergent.} \end{cases}$

(ii) $\sum \frac{1}{n} x^n$: $\rho = \limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$ (Ex!)

$\Rightarrow R = \frac{1}{\rho} = 1$

$\begin{cases} x=1: & \sum \frac{1}{n} x^n = 1+\frac{1}{2}+\frac{1}{3}+\cdots \text{ is divergent} \\ x=-1: & \sum \frac{1}{n} x^n = 1-\frac{1}{2}+\frac{1}{3}-\cdots \text{ is convergent.} \end{cases}$

(iii) $\sum \frac{1}{n^2} x^n$: $\rho = \limsup |a_n|^{\frac{1}{n}} = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = 1$ (Ex!)

$\Rightarrow R = \frac{1}{\rho} = 1$

$\begin{cases} x=1: & \sum \frac{1}{n^2} x^n = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \text{ is convergent.} \\ x=-1: & \sum \frac{1}{n^2} x^n = 1-\frac{1}{2^2}+\frac{1}{3^2}-\cdots \text{ is convergent.} \end{cases}$

Pf of Cauchy-Hadamard Thm:

- $R=0$ and $R=+\infty$ leave as exercises.

Assume $0<R<+\infty$.

Clearly $\sum a_n x^n$ converges for $x=0$.

Consider $0<|x|<R$,

then $\exists\ 0<c<1$ such that $|x|<cR \ (=\frac{c}{\rho})$

Therefore $\rho|x| = \limsup\left(|a_n|^{\frac{1}{n}}|x|\right) < c$.

$\Rightarrow \exists K\in\mathbb{N}$ such that

if $n\geq K$, then $|a_n|^{\frac{1}{n}}|x| \leq c$.

$$\Rightarrow \quad |a_n x^n| \leq c^n, \ \forall n\geq K$$

Since $0<c<1$, $\sum c^n$ is convergent.

By Comparison Test (Thm 3.7.7), $\sum |a_n x^n|$ is convergent

i.e. $\sum a_n x^n$ is absolutely convergent.

This proves the 1st part.

If $|x|>R=\frac{1}{\rho}$, then $\rho = \limsup |a_n|^{\frac{1}{n}} > \frac{1}{|x|}$.

$\Rightarrow |a_n|^{\frac{1}{n}} > \frac{1}{|x|}$ for infinitely many $n\in\mathbb{N}$

i.e. $|a_n x^n|>1$ for infinitely many $n\in\mathbb{N}$

and hence $a_n x^n \not\to 0$. $\therefore \sum a_n x^n$ is divergent. ⋇

Remarks: (i) If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, then radius of convergence $= \lim \left|\frac{a_n}{a_{n+1}}\right|$.

(Ex 9.4.5)

(ii) If one can choose $0<c<1$ independent of $x \in (-R,R)$, then one get uniform convergence.

Thm 9.4.10: Let
- $R=$ radius of convergence of $\sum a_n x^n$
- $[a,b] \subset (-R,R)$ be a closed and bounded interval.

Then $\sum a_n x^n$ converges uniformly on $[a,b]$.

Remark
- $R=+\infty$ included, and hence we need the assumption that $[a,b]$ is bounded.
- $R=0$ is excluded as $(-0,0)=\emptyset$. (althought $\sum a_n x^n$ converges for $x=0$)

Pf of Thm 9.4.10: Since $[a,b] \subset (-R,R)$, $\exists\ 0<c<1$ such that

$-cR<a$ and $b<cR$. (Note: c depends only on a,b)

Therefore $\forall\ x \in [a,b]$, $|x|<cR$.

By argument in the proof of Cauchy-Hadamard Thm, we have $\exists\ K \in \mathbb{N}$ s.t. $|a_n x^n| \leq c^n$, $\forall\ n \geq K$ (Ex! use a $0<c_1<c$ s.t. $-c_1R<a<b<c_1R$ to find a K indep. of x)

Since $\sum c^n$ is convergent, Weierstrass M-Test (Thm 9.4.6)

$\Rightarrow$ ($\sum_{n=K}^{\infty} a_n x^n$ and hence) $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[a,b]$. #

Thm 9.4.11

- The limit of power series is continuous on the interval of convergence.
- A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.

Pf: • $\forall\, x \in (-R, R)$, choose a closed & bounded interval $[a,b]$ s.t. $x \in [a,b] \subset (-R,R)$. Then on $[a,b]$,

$\sum a_n x^n$ converges uniformly.

Thm 9.4.2 $\Rightarrow$ $\sum_{n=1}^{\infty} a_n x^n$ is continuous on $[a,b]$ and hence at x

Since $x \in (-R,R)$ is arbitrary, $\sum_{n=0}^{\infty} a_n x^n$ is continuous on $(-R,R)$.

• For any closed and bounded interval $[a,b] \subset (-R,R)$,

$\sum a_n x^n$ converges uniformly on $[a,b]$ (Thm 9.4.10)

and hence Thm 9.4.3 $\Rightarrow$

$$\int_a^b \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \int_a^b a_n x^n .$$

$\#$

Thm 9.4.12 (Differentiation Thm)

A power series can be differentiated term-by-term within the interval of convergence. In fact, if R = radius of convergence of $\Sigma a_n x^n$ and $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$,

then the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^n = R$,

and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, for $|x| < R$

Pf: Since $n^{\frac{1}{n}} \to 1$, the seq. $\left(|(n+1) a_{n+1}|^{\frac{1}{n+1}}\right)$ is bounded

$\Leftrightarrow$ the seq $\left(|a_n|^{\frac{1}{n}}\right)$ is bounded

$\therefore$ (unbounded)

- $R = 0 \Leftrightarrow$ Radius of convergence of $\Sigma n a_n x^n = 0$

(bounded)

- Radius of convergence of $\Sigma n a_n x^n = \limsup |(n+1) a_{n+1}|^{\frac{1}{n+1}}$

$= \limsup |n a_n|^{\frac{1}{n}} = \limsup \left(n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}\right)$

$= \limsup |a_n|^{\frac{1}{n}}$ (since $n^{\frac{1}{n}} \to 1$)

$= R$.

Hence Radius of convergence of $\Sigma n a_n x^n$

= Radius of convergence of $\Sigma a_n x^n$.

Note that $\Sigma a_n x^n$ converges for $x = 0$

Now $\forall x \in (-R,R)$, choose $0 < a < R$ such that $|x| < a$.

Then • $[-a,a]$ is closed and bounded,

• $[-a,a] \subset (-R,R)$ and

• $0 \in [-a,a]$ s.t. $\sum a_n x^n$ converges at $x=0$.

Using Thm 9.4.10, Thm 8.2.3 and note that

• $(a_n x^n)' = n a_n x^{n-1}$

• $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} (a_n x^n)'$ converges uniformly on $[-a,a]$,

we have $$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ on } [-a,a]$$

and in particular for x.

Since $x \in (-R,R)$ is arbitrary, we have

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \forall x \in (-R,R)$$

✕✕

Remarks: (i) Differentiation Thm 9.4.12 makes no conclusion for $|x| = R$:

e.g. $\sum \frac{1}{n^2} x^n$ converges for $|x| = 1$ $(=R)$

but $\left(\sum \frac{1}{n^2} x^n\right)' = \sum \frac{1}{n} x^{n-1}$ $\begin{cases} \text{converges at } x=-1 \\ \text{diverges at } x=1. \end{cases}$

(ii) Repeated application of Thm 9.4.12 $\Rightarrow$ $\forall k \in \mathbb{N}$,

$$\boxed{\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}}$$

Thm 9.4.13 (Uniqueness Thm)

If $\sum a_n x^n$ & $\sum b_n x^n$ converge to the same function f on an interval $(-r, r)$, $r>0$, then

$$a_n = b_n, \quad \forall n \in \mathbb{N}$$

(In fact $a_n = b_n = \frac{1}{n!} f^{(n)}(0)$)

Pf: By remark (ii) of Thm 9.4.12, $\forall k \in \mathbb{N}$,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} \quad \forall x \in (-r, r).$$

$$\Rightarrow \quad f^{(k)}(0) = \frac{k!}{(k-k)!} a_k \quad (0^{n-k} = 0 \text{ for } n > k)$$

$$\Rightarrow \quad a_k = \frac{1}{k!} f^{(k)}(0)$$

Same for b_k. ※

Taylor Series

Let f has derivatives of all orders at a point $c \in \mathbb{R}$, then one can form a power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n.$$

Note that
- no convergence yet (unless $x=c$)
- Even it converges, it may not equal f (Ex. 9.4.12)

Def We say that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

is the <u>Taylor expansion</u> of f at c if $\exists R>0$ such that

$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ converges to $f(x)$ on $(c-R, c+R)$,

and $\frac{f^{(n)}(c)}{n!}$ are called <u>Taylor coefficients</u>.

(i.e. The remainder $R_n(x)$ in Taylor's Thms $\to 0$ on $(c-R, c+R)$)

<u>Remark</u>: By Uniqueness Thm 9.4.13, if Taylor expansion exists, it is unique.

<u>Eg 9.4.14</u>

(a) $f(x) = \sin x$, $x \in \mathbb{R}$,

Then $f^{(n)}(x) = \begin{cases} (-1)^k \sin x, & \text{if } n=2k \\ (-1)^k \cos x, & \text{if } n=2k+1. \end{cases}$

At $c=0$, we have $f^{(n)}(0) = \begin{cases} 0, & \text{if } n=2k \\ (-1)^k, & \text{if } n=2k+1 \end{cases}$

Furthermore, by Taylor's Thm 6.4.1, the remainder $R_n(x)$ satisfies

$$|R_n(x)| = \frac{|f^{(n+1)}(c_1)|\,|x|^{n+1}}{(n+1)!} \quad \text{for some } c_1 \text{ between } x \,\&\, 0$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \to 0$$

$\therefore \quad \sin x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \forall x \in \mathbb{R}$

is the Taylor expansion of $\sin x$ at $x=0$.

Then application of Differentiation Thm 9.4.12, we have

$$\cos x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \forall x \in \mathbb{R}$$

is the Taylor expansion of $\cos x$ at $x=0$.

(b) $g(x) = e^x, \quad x \in \mathbb{R}$

Then $g^{(n)}(x) = e^x, \ \forall x \in \mathbb{R} \Rightarrow g^{(n)}(0) = 1$.

By Taylor's Thm 6.4.1, the remainder satisfies

$$|R_n(x)| \le \frac{e^c}{(n+1)!} |x|^{n+1} \quad \text{for some } c \text{ between } x \ \& \ 0.$$

$$\le \frac{e^{|x|} |x|^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n \to \infty.$$

$\therefore \quad e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \forall x \in \mathbb{R}$

is the Taylor expansion of e^x at $x=0$.

Furthermore, by $e^x = e^c e^{x-c} = e^c \sum_{n=0}^{\infty} \frac{1}{n!}(x-c)^n$,

we see that $e^x = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n$ is the

Taylor expansion of e^x at $x=c$. #

Remark: This implies the radius of convergence $= +\infty$ (says at $c=0$). Of course, one can derive it from calculating $\left(\frac{1}{n!}\right)^{\frac{1}{n}} \to 0$ as $n \to +\infty$.