The Integral Test

Def (Improper Integral)

For $a \in \mathbb{R}$, if $\begin{cases} \bullet\ f \in R[a,b],\ \forall b > a, \text{ and} \\ \bullet\ \lim\limits_{b\to+\infty} \int_a^b f \text{ exists (and } < +\infty.) \end{cases}$

then the improper integral $\int_a^\infty f$ is defined to be

$$\int_a^\infty f = \lim_{b\to+\infty} \int_a^b f .$$

Thm 9.2.6 (Integral Test)

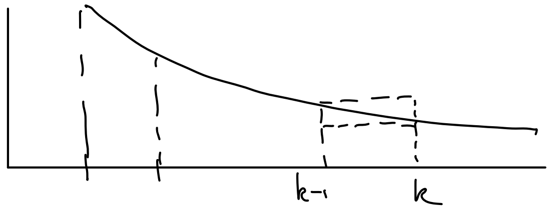
Let $f(t) > 0$, decreasing on $\{t \geq 1\}$.

Then $\sum\limits_{k=1}^{\infty} f(k)$ converges $\Leftrightarrow$ $\int_1^\infty f = \lim\limits_{b\to+\infty} \int_1^b f$ exists.

In this case,

$$\int_{n+1}^{\infty} f(t)\,dt \leq \sum_{k=1}^{\infty} f(k) - \sum_{k=1}^{n} f(k) \leq \int_n^\infty f(t)\,dt, \quad \forall n = 1, 2, \cdots$$

Pf:



$f > 0$ & decreasing $\Rightarrow$ $\forall k = 2, 3, \cdots$

$$f(k) \leq \int_{k-1}^{k} f(t)\,dt \leq f(k-1) \qquad — (*)_1$$

$$\Rightarrow \sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(t)\,dt \le \sum_{k=2}^{n} f(k-1) = f(1)+\cdots+f(n-1)$$

Let $S_n = \sum_{k=1}^{n} f(k)$

Then, we have

$$S_n - f(1) \le \int_1^n f(t)\,dt \le S_{n-1}.$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n \text{ exists} \Rightarrow \lim_{n\to\infty} \int_1^n f(t)\,dt \text{ exists}$$

$$\therefore \quad \sum_{k=1}^{\infty} f(k) \text{ converges} \Leftrightarrow \int_1^\infty f \text{ exists}.$$

Using $(*)_1$ again, if $m>n$, then

$$\sum_{k=n+1}^{m} f(k) \le \sum_{k=n+1}^{m} \int_{k-1}^{k} f(t)\,dt \le \sum_{k=n+1}^{m} f(k-1)$$

$$\Rightarrow \quad S_m - S_n \le \int_n^m f(t)\,dt \le S_{m-1} - S_{n-1}$$

Hence, $\forall\, m>n$, we have

$$\int_{n+1}^{m+1} f(t)\,dt \le S_m - S_n \le \int_n^m f(t)\,dt$$

Letting $m\to\infty$, we have

$$\int_{n+1}^{\infty} f(t)\,dt < S - S_n \le \int_n^\infty f(t)\,dt$$

where $S = \sum_{k=1}^{\infty} f(k)$. ※

Egs 9.2.7

(a) Recall Eg 3.7.2(c)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$$ is $\overbrace{\text{convergent}}^{\text{(absolutely)}}$ $\left(\frac{1}{n(n+1)} > 0\right)$.

Using Limit Comparison Test II (Thm 9.2.1)

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$$

$\Rightarrow$ $\sum \frac{1}{n^2}$ is (absolutely) convergent.

(b) However, Root Test (Thm 9.2.2) doesn't apply to $\sum \frac{1}{n^2}$

(in fact $\sum \frac{1}{n^p}$, $\forall p > 0$):

$$\begin{cases} \bullet \ \left(\frac{1}{n^p}\right)^{\frac{1}{n}} < 1, \text{ and} \\ \bullet \ \left|\frac{1}{n^p}\right|^{1/n} = \frac{1}{(n^{\frac{1}{n}})^p} \to 1 \quad \text{since } n^{\frac{1}{n}} \to 1 \end{cases}$$

$\therefore$ both conditions in part (a) & part (b) don't hold.

And the Cor 9.2.3 cannot be applied too.

$$\left(r = \lim_{n\to\infty} \left|\frac{1}{n^p}\right|^{\frac{1}{n}} = 1 . \right)$$

(c) Ratio Test (Thm 9.2.4) and its Cor 9.25 also don't work for $\sum \frac{1}{n^p}$:

$$\left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right| = \frac{n^p}{(n+1)^p} = \frac{1}{(1+\frac{1}{n})^p} \to 1$$

$r=1$, no information from Ratio test!

(d) On the other hand, Integral Test (Thm 9.2.6) works for $\sum \frac{1}{n^p}$:

Let $f(t) = \frac{1}{t^p}$, $t \geq 1$.

Then $f(t) > 0$ and decreasing.

$$\lim_{n\to\infty} \int_1^n \frac{1}{t^p}\,dt = \begin{cases} \lim_{n\to\infty} (\ln(n) - \ln 1), & p=1 \\ \lim_{n\to\infty} \left[\frac{t^{1-p}}{1-p}\right]_1^n, & p \neq 1 \end{cases}$$

(same as $b\to\infty$)

Since $\ln(n) \to +\infty$, $\int_1^\infty \frac{1}{t}\,dt$ doesn't exist

$$\lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right) = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ +\infty, & \text{if } p<1 \end{cases}$$

$\therefore \int_1^\infty \frac{1}{t^p}\,dt$ $\begin{cases} \text{exists} & \text{if } p>1 \\ \text{doesn't exist} & \text{if } p<1. \end{cases}$

Altogether, $\sum \frac{1}{n^p}$ $\begin{cases} \text{converges} & \text{if } p>1 \\ \text{diverges} & \text{if } p\leq 1 \end{cases}$

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Thm 9.2.8 (Raabe's Test) Suppose $x_n \neq 0$, $\forall n = 1, 2, 3, \cdots$

(a) If $\exists\ a > 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}, \quad \forall n \geq K$$

(Note: This condition allows $\lim \left|\frac{x_{n+1}}{x_n}\right| = 1$)

then $\sum x_n$ is absolutely convergent

(b) If $\exists\ a \leq 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n} \quad \forall n \geq K$$

(Note: This condition allows $\lim \left|\frac{x_{n+1}}{x_n}\right| = 1$)

then $\sum x_n$ is not absolutely convergent.

Pf: (a) The condition $\Rightarrow$ $\forall n \geq K$

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n}$$

$$\therefore \quad n|x_{n+1}| \leq (n-a)|x_n| = ((n-1)-(a-1))|x_n|$$

$$= (n-1)|x_n| - (a-1)|x_n|$$

$$\Rightarrow \quad 0 < (a-1)|x_n| \leq (n-1)|x_n| - n|x_{n+1}|, \quad \forall n \geq K. \quad \text{—— } (*)$$

$\therefore$ $(y_n) = (n|x_{n+1}|)$ is a decreasing sequence for $n \geq K$

Summing $(*)$ for $n = K, \cdots, m$, we have

$$0 < (a-1)\sum_{n=K}^{m} |x_n| \leq \sum_{n=K}^{m} (y_{n-1} - y_n) = y_{K-1} - y_m$$

$$= (K-1)|x_K| - m|x_{m+1}| < (K-1)|x_K|$$

$\therefore$ $\sum |x_n|$ is bounded

This implies $\sum |x_n|$ is convergent,

hence $\sum x_n$ is absolutely convergent.

(b) $\left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n}$, $\forall n \geq K$ $(a \leq 1)$

$\Rightarrow$ $n|x_{n+1}| \geq (n-a)|x_n| \geq (n-1)|x_n|$ since $a \leq 1$

$\therefore$ $(n|x_{n+1}|)$ is increasing $\forall n \geq K$.

$\Rightarrow$ $n|x_{n+1}| \geq K|x_{K+1}|$, $\forall n \geq K$

i.e. $|x_{n+1}| \geq \frac{c}{n}$, $\forall n \geq K$ where $c = K|x_{K+1}|$ is a constant.

Hence $\sum |x_n|$ diverges, since $\sum \frac{1}{n}$ diverges.

$\therefore$ $\sum x_n$ is not absolutely convergent. ※

Cor 9.2.9 $\left\{ \begin{array}{l} \bullet\ x_n \neq 0,\ \forall n = 1, 2, 3, \cdots \\ \bullet\ a = \lim_{n \to \infty} n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) \text{ exists} \end{array} \right.$

Then $\left\{ \begin{array}{l} \bullet\ a > 1 \Rightarrow \sum x_n \text{ is absolutely convergent} \\ \bullet\ a < 1 \Rightarrow \sum x_n \text{ is not absolutely convergent} \end{array} \right.$

Pf: If $a > 1$, then $\forall a_1$ with $1 < a_1 < a$, $\exists K \in \mathbb{N}$ s.t.

$$n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) \geq a_1, \quad \forall n \geq K$$

$\Rightarrow$ $\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a_1}{n}$, $\forall n \geq K$

$\therefore$ Thm 9.2.8 $\Rightarrow$ $\sum x_n$ is absolutely convergent.

If $a<1$, then $\forall\ a_1$ with $a<a_1\leq 1$, $\exists K\in\mathbb{N}$ s.t.

$$n\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right)\leq a_1,\quad \forall\ n\geq K$$

$$\Rightarrow \qquad \left|\frac{x_{n+1}}{x_n}\right|\geq 1-\frac{a_1}{n},\quad \forall\ n\geq K$$

$\therefore$ Thm 9.2.8 $\Rightarrow$ $\sum x_n$ is not absolutely convergent

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<u>Egs 9.2.10</u>

(a) Raabe's Test for $\sum \frac{1}{n^p}$:

$$a=\lim_{n\to\infty} n\left(1-\left|\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}}\right|\right)=\lim_{n\to\infty} n\left(1-\frac{n^p}{(n+1)^p}\right)$$

$$=\lim_{n\to\infty} n\left(1-\frac{1}{(1+\frac{1}{n})^p}\right)=\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^p-1}{\frac{1}{n}}\cdot\frac{1}{(1+\frac{1}{n})^p}\right]$$

Clearly $\displaystyle\lim_{n\to\infty}\frac{(1+\frac{1}{n})^p-1}{\frac{1}{n}}=\frac{d}{dx}\Big|_{x=1} x^p=p$ (Thm 8.3.13)

$\therefore\quad a=p\cdot 1=p$.

By Cor 9.2.9 to Raabe's Test (or just call it Raabe's Test),

$p>1\Rightarrow \sum\frac{1}{n^p}$ is (absolutely) convergent

$p<1\Rightarrow \sum\frac{1}{n^p}$ is not (absolutely) convergent

(hence divergent (as $\frac{1}{n^p}>0,\ \forall n$))

However, result for $p=1$ cannot be deduced from Raabe's Test.

(b) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Easy to check:

$$\begin{cases} \bullet \ \left|\frac{x_{n+1}}{x_n}\right| = \dfrac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \frac{n+1}{n} \cdot \frac{n^2+1}{(n+1)^2+1} \to 1, \text{ and} \\ \bullet \ n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) = n \cdot \left(1 - \frac{n+1}{n} \frac{n^2+1}{(n+1)^2+1}\right) \end{cases}$$

$$= \frac{n^2+n-1}{(n+1)^2+1} \to 1 \quad \text{as } n \to \infty$$

$\therefore$ Both Cor 9.2.5 and Cor 9.2.2 cannot be applied.

But $\left|\frac{x_{n+1}}{x_n}\right| - 1 = \frac{n+1}{n} \frac{n^2+1}{(n+1)^2+1} - 1 = \frac{(n+1)(n^2+1) - n[(n+1)^2+1]}{n[(n+1)^2+1]}$

$$= -\frac{n^2+n-1}{n[(n+1)^2+1]} = -\frac{1}{n} \cdot \frac{n^2+n-1}{n^2+2n+2} \geq -\frac{1}{n}$$

$$\therefore \quad \left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{1}{n}, \quad \forall n \geq 1 \quad \left(\begin{matrix} a = 1 \leq 1 \\ \& \ K = 1 \in \mathbb{N} \end{matrix}\right)$$

Raabe's Test (Thm 9.2.8) $\Rightarrow$ $\sum x_n$ is not absolutely convergent.

Remarks: (i) "Limiting form" of Raabe's Test (Cor 9.2.9) doesn't apply, but Raabe's Test (Thm 9.2.8) applies.

(ii) Integral Test or Limit Comparison Test work for this example.

§9.3 Tests for Nonabsolute Convergence

Def 9.3.1 • $x_n \neq 0$, $\forall n \in \mathbb{N}$

Then • the sequence (x_n) is said to be alternating

if $(-1)^{n+1} x_n > 0$ (or < 0) $\forall n \in \mathbb{N}$

• in this case, the series $\sum x_n$ is called an

alternating series.

eg. If $z_n > 0$, then $x_n = (-1)^{n+1} z_n$ and $x_n = (-1)^n z_n$ are alternating.

(explicit eg: $z_n = \frac{1}{n} > 0$, $(x_n) = \left((-1)^{n+1} z_n\right) = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots)$ is alternating)

Thm 9.3.2 Let • $z_n > 0$ and decreasing $(z_{n+1} \leq z_n)$ $\forall n \in \mathbb{N}$

• $\lim_{n \to \infty} z_n = 0$

Then the alternating series $\sum (-1)^{n+1} z_n$ is convergent

Pf: Consider partial sum

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k+1} z_k = z_1 - z_2 + z_3 - z_4 + \cdots + z_{2n-1} - z_{2n}$$

Then $S_{2(n+1)} - S_{2n} = z_{2n+2} - z_{2n+1} \geq 0$,

since z_n is decreasing

$\therefore$ (S_{2n}) is increasing (in n).

Also $z_1 - S_{2n} = \underbrace{z_2 - z_3}_{\geq 0} + \underbrace{z_4 - z_5}_{\geq 0} + \cdots + \underbrace{z_{2n-2} - z_{2n-1}}_{\geq 0} + \underbrace{z_{2n}}_{> 0} > 0$

$\therefore$ (S_{2n}) is bounded above by z_1

By Monotone Convergence Thm (Thm 3.3.2), $\exists\, S \in \mathbb{R}$ s.t.

$$S_{2n} \to S \quad \text{as } n \to \infty.$$

Together with $z_n \to 0$ as $n \to \infty$, we have

$\forall\, \varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t.

if $n \geq K$, then
- $|S_{2n} - S| < \frac{\varepsilon}{2}$, and
- $(0 <)\, z_{2n+1} < \frac{\varepsilon}{2}$.

Then $|S_{2n+1} - S| = |z_{2n+1} + S_{2n} - S|$

$$\leq |z_{2n+1}| + |S_{2n} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ $S_{2n+1} \to S$ as $n \to \infty$.

Combining with $S_{2n} \to S$ as $n \to \infty$, we have

$$\lim S_n = S$$

$\therefore$ $\sum (-1)^{n+1} z_n$ is convergent. ※

egs: By Thm 9.3.1, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

is convergent

(Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ is divergent by integral Test eg 9.2.7 (d))

The Dirichlet and Abel Tests

Thm 9.3.3 (Abel's Lemma)

Let
- (x_n), (y_n) be sequences in $\mathbb{R}$, and
- $\begin{cases} s_0 = 0, & \& \\ s_n = \sum_{k=1}^{n} y_k, & n = 1, 2, 3 \end{cases}$

Then for $m > n$,

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Pf: $\displaystyle\sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$

$$= x_m (s_m - s_{m-1}) + x_{m-1}(s_{m-1} - s_{m-2}) + \cdots + x_{n+1}(s_{n+1} - s_n)$$

$$= x_m s_m - (x_m - x_{m-1}) s_{m-1} - (x_{m-1} - x_{m-2}) s_{m-2} - \cdots - (x_{n+2} - x_{n+1}) s_{n+1} - x_{n+1} s_n$$

$$= (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$$

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