(Cont'd) Now by Fundamental Thm of Calculus,

$$C_n'(x) = -S_{n-1}(x) \underset{\text{(uniformly)}}{\rightrightarrows} -S(x) \text{ on } [-A,A], \ \forall A>0$$

Thm 8.2.3 $\Rightarrow$

$$C(x) = \lim_{n\to\infty} C_n(x) \text{ is differentiable and}$$

$$C'(x) = -S(x) \qquad \text{on } [-A,A], \ \forall A>0$$

Hence C is differentiable $\forall x \in \mathbb{R}$ and

$$C'(x) = -S(x), \ \forall x \in \mathbb{R}.$$

In particular, $C'(0) = -S(0) = 0$

Similarly, Fundamental Thm

$$\Rightarrow S_n'(x) = C_n(x) \rightrightarrows C(x) \text{ on } [-A,A], \ \forall A>0$$

$\vdots$ (Ex!)

$\Rightarrow$ S is differentiable $\forall x \in \mathbb{R}$ &

$$S'(x) = C(x), \ \forall x \in \mathbb{R}.$$

In particular, $S'(0) = C(0) = 1$.

Finally, combining the 2 formulae of 1st derivatives, we have

$$C''(x) = -S'(x) = -C(x) \quad \&$$

$$S''(x) = C'(x) = -S(x).$$

※

Cor 8.4.2 If C, S are the functions in Thm 8.4.1, then

(iii) $C'(x) = -S(x)$ & $S'(x) = C(x)$, $\forall x \in \mathbb{R}$.

Moreover, C & S have derivatives of all orders

Pf: (iii) is included in the proof of Thm 8.4.1,

The last statement follows easily by induction.

Cor 8.4.3 The functions C & S in Thm 8.4.1 satisfy the Pythagorean Identity: $(C(x))^2 + (S(x))^2 = 1$, $\forall x \in \mathbb{R}$

Pf: Let $f(x) = (C(x))^2 + (S(x))^2$.

By Thm 8.4.1, f is differentiable &

$$f'(x) = 2C(x)C'(x) + 2S(x)S'(x)$$

$$= -2C(x)S(x) + 2S(x)C(x) = 0, \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ $f(x)$ is a constant function on $\mathbb{R}$.

$\Rightarrow$ $f(x) \equiv f(0) = (C(0))^2 + (S(0))^2 = 1$, $\forall x \in \mathbb{R}$. ※

Thm 8.4.4 The functions C and S satisfying

$$(*)_C \begin{cases} C'' = -C \\ C(0) = 1 \\ C'(0) = 0 \end{cases} \quad \text{and} \quad (*)_S \begin{cases} S'' = -S \\ S(0) = 0 \\ S'(0) = 1 \end{cases}$$

are unique.

Pf: Let C_1 & C_2 satisfy $(*)_C$, and S_1 & S_2 satisfy $(*)_S$.

Define $D = C_1 - C_2$

$T = S_1 - S_2$

Then D satisfies $D'' = C_1'' - C_2'' = (-C_1) - (-C_2) = -D$

& $D' = C_1' - C_2' = -S_1 - (-S_2) = S_2 - S_1 = -T$

Hence D has derivatives of all order and

$D(0) = C_1(0) - C_2(0) = 0$,

$D'(0) = C_1'(0) - C_2'(0) = 0$

$D''(0) = -D(0) = 0$

$D^{(3)}(0) = -D'(0) = 0$

$\vdots$

$D^{(k)}(0) = 0$, $\forall k$ (by induction)

Now $\forall x \in \mathbb{R}$, consider closed interval I_x with end points x & 0.

Then $\exists K>0$ s.t.

$$|D(t)| \le K \ \& \ |T(t)| \le K, \ \forall t \in I_x$$

since both $D = c_1 - c_2$ & $T = s_1 - s_2$ are continuous.

Applying Taylor's Thm 6.4.1, we have

$$D(x) = D(0) + \frac{D'(0)}{1!}x + \cdots \frac{D^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{D^{(n)}(c_n)}{n!}x^n$$

for some $c_n \in I_x$.

Using $D^{(n-1)}(0) = \cdots = D(0) = 0$, we have

$$D(x) = \frac{D^{(n)}(c_n)}{n!}x^n$$

Note that if $n = 2k$, $|D^{(n)}(c_n)| = |D(c_n)| \le K$

if $n = 2k+1$, $|D^{(n)}(c_n)| = |D'(c_n)| = |T(c_n)| \le K$

$$\therefore \quad |D(x)| \le \frac{K}{n!}|x|^n$$

Since $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$, we have $D(x) = 0$.

As $x \in \mathbb{R}$ is arbitrary, we've proved that

$$c_1(x) = c_2(x) \quad \forall x \in \mathbb{R}.$$

Similarly, one can prove $s_1(x) = s_2(x)$, $\forall x \in \mathbb{R}$

$\#$

Def 8.4.5 The unique functions C & S given in Thm 8.4.1 are called the <u>cosine function</u> and the <u>sine function</u> respectively, and denoted by

$$\cos x = C(x) \quad \& \quad \sin x = S(x)$$

Thm 8.4.6: If $f:\mathbb{R}\to\mathbb{R}$ satisfies $f''(x) = -f(x)$, $\forall x\in\mathbb{R}$, then $\exists$ real numbers α, β such that

$$f(x) = \alpha C(x) + \beta S(x), \quad \forall x \in \mathbb{R}.$$

Pf: Let $\alpha = f(0)$ & $\beta = f'(0)$.

And consider $g(x) = \alpha C(x) + \beta S(x)$, $\forall x\in\mathbb{R}$.

Then
- $g(0) = \alpha C(0) + \beta S(0) = \alpha = f(0)$
- $g'(x) = -\alpha S(x) + \beta C(x)$

  $\Rightarrow g'(0) = -\alpha S(0) + \beta C(0) = \beta = f'(0)$.
- $g''(x) = \alpha C''(x) + \beta S''(x) = -g(x)$

Hence the function $h = f - g$ satisfies

$$\begin{cases} h'' = f'' - g'' = -f - (-g) = -h \\ h(0) = f(0) - g(0) = 0 \\ h'(0) = f'(0) - g'(0) = 0 \end{cases}$$

Similarly argument as in the proof of Thm 8.4.4,

we have $h(x) = 0, \ \forall x \in \mathbb{R}$.

$\therefore \quad f(x) = g(x) = \alpha C(x) + \beta S(x) \quad \forall x \in \mathbb{R}$. #

Thm 8.4.7 The cosine $C(x)$ & sine $S(x)$ satisfy

(v) $C(-x) = C(x)$ & $S(-x) = -S(x) \quad \forall x \in \mathbb{R}$

(vi) $\begin{cases} C(x+y) = C(x)C(y) - S(x)S(y) \\ S(x+y) = S(x)C(y) + C(x)S(y) \end{cases}$ (compound angle formulae)

Pf: (v) Let $\varphi(x) = C(-x)$ (Similarly for $S(x)$ (Ex!))

Then $\varphi''(x) = C''(-x) = -C(-x) = -\varphi(x)$

$\varphi(0) = C(0) = 1$

$\varphi'(x) = -C'(-x) \Rightarrow \varphi'(0) = -C'(0) = 0$

By uniqueness Thm 8.4.4, $\varphi(x) = C(x), \ \forall x \in \mathbb{R}$ #

(vi) Fix $y \in \mathbb{R}$, and let

$f(x) = C(x+y)$

Then
- $f(0) = C(y)$
- $f'(0) = C'(y) = -S(y)$.
- $f''(x) = C''(x+y) = -C(x+y) = -f(x)$

By Thm 8.4.6 and its proof,

$\exists\ \alpha = f(0),\ \beta = f'(0)$ such that

$$f(x) = \alpha C(x) + \beta S(x) = C(y)C(x) - S(y)S(x), \quad \forall x \in \mathbb{R}$$

Since $y \in \mathbb{R}$ is arbitrary,

$$C(x+y) = C(x)C(y) - S(x)S(y), \quad \forall x, y \in \mathbb{R}.$$

Differentiate wrt x, we have

$$C'(x+y) = C'(x)C(y) - S'(x)S(y)$$

$$-S(x+y) = -S(x)C(y) - C(x)S(y)$$

$$\Rightarrow \quad S(x+y) = S(x)C(y) + C(x)S(y), \quad \forall x, y \in \mathbb{R}$$

#

<u>Thm 8.4.8</u> For $x \geq 0$,

(vii) $-x \leq S(x) \leq x$; (viii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$; (x) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

<u>Pf</u>: Cor 8.4.3 : $C(t)^2 + S(t)^2 = 1$

$$\Rightarrow \quad -1 \leq C(t) \leq 1.$$

$$\Rightarrow \forall x \geq 0, \quad -x \leq \int_0^x C(t)\,dt \leq x$$

By Fundamental Thm, there is

$$-x \leq S(x) \leq x, \quad \text{ie (vii)}$$

Integrate again $\quad -\frac{1}{2}x^2 \leq \int_0^x S(t)\,dt \leq \frac{1}{2}x^2$

Fundamental Thm $\Rightarrow \quad -\frac{1}{2}x^2 \leq -C(x) + C(0) \leq \frac{1}{2}x^2$

$$\Rightarrow \quad 1 - \frac{1}{2}x^2 \leq C(x) \quad (\quad)$$

Together with $-1 \le C(x) \le 1$, we have (viii).

Further integrating (viii), we have

$$x - \frac{x^3}{6} \le S(x) \le x \quad . \qquad (ix)$$

integrating again $\quad \frac{x^2}{2} - \frac{x^4}{24} \le -C(x) + 1 \le \frac{x^2}{2}$

$$\Rightarrow \qquad 1 - \frac{x^2}{2} \le C(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \#$$

<u>Lemma 8.4.9</u> • $\exists$ a root γ of $C(x)$ in the interval $[\sqrt{2}, \sqrt{3})$.

- Moreover, $C(x) > 0 \ \forall x \in [0, \gamma)$.
- The number 2γ is the smallest positive root of $S(x)$.

<u>Pf</u>: By ineq. (x) in Thm 8.4.8

$$1 - \frac{1}{2}x^2 \le C(x) \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

we have $\quad C(\sqrt{2}) \ge 0 \quad$ and

$$C(\sqrt{3}) \le 1 - \frac{1}{2}(\sqrt{3})^2 + \frac{1}{24}(\sqrt{3})^4$$
$$= 1 - \frac{3}{2} + \frac{9}{24} = \frac{24 - 36 + 9}{24} = -\frac{1}{8} < 0$$

Intermediate value Thm $\Rightarrow$ $C(x) = 0$ for some $x \in [\sqrt{2}, \sqrt{3})$.

Let γ be the <u>smallest</u> such root of $C(x)$ in $[\sqrt{2}, \sqrt{3})$.

Then $\forall x \in [0, \gamma)$, if $x \in [\sqrt{2}, \gamma)$, then $C(x) \ne 0$ by the choice of γ.

If $x \in [0, \sqrt{2})$, then $C(x) \geq 1 - \frac{1}{2}x^2 > 0$.

Therefore, continuity of $C(x) \Rightarrow C(x) > 0, \forall x \in [0, \gamma)$

Finally by Thm 8.4.7 (with $x=y$), $S(2x) = 2S(x)C(x)$

Therefore $S(2\gamma) = 2S(\gamma)C(\gamma) = 0$

$\therefore$ 2γ is a positive root of $S(x)$.

Now let 2δ = smallest positive root of $S(x)$.

(using $S(0)=0$ & $S'(0)=1$)

Suppose $\delta < \gamma$

Then $0 = S(2\delta) = 2S(\delta)C(\delta)$,

Since $C(x) > 0, \forall x \in [0, \gamma)$, we have

$$S(2 \cdot \tfrac{\delta}{2}) = S(\delta) = 0$$

which contradicts the definition of δ.

Therefore $\delta = \gamma$. ※

Note: Of course, we can prove that $\gamma > \sqrt{2}$ as stated in the Textbook. But we need Ex 8.4.4 (not just Thm 8.4.8).

Def 8.4.10 $\quad \pi \overset{\text{def}}{=} 2\gamma$ = smallest positive root of S

Note: Thm 8.4.8 (x) $\Rightarrow$ $2.828 \leq \pi \leq 2 \times \underbrace{\sqrt{6 - 2\sqrt{3}}}_{}$ < 3.185 (Ex!)

smallest positive root of $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Thm 8.4.11

- C & S are <u>2π-periodic</u> (have <u>period</u> 2π)

(xi) $C(x+2\pi) = C(x)$ & $S(x+2\pi) = S(x)$, $\forall x \in \mathbb{R}$

- $\begin{cases} S(x) = C(\frac{\pi}{2}-x) = -C(x+\frac{\pi}{2}) \\ C(x) = S(\frac{\pi}{2}-x) = S(x+\frac{\pi}{2}) \end{cases}$ $\forall x \in \mathbb{R}$

<u>Pf of (xi):</u> By $\begin{cases} S(2x) = 2S(x)C(x) \text{ \&} \\ S(\pi) = 0 \end{cases}$

we have $S(2\pi) = 0$.

On the other hand, Cor 8.4.3 $\Rightarrow$ $1 = C(\pi)^2 + S(\pi)^2 = C(\pi)^2$

which combining with Thm 8.4.7 (for cases $x=y$) $\Rightarrow$

$$C(2\pi) = C(\pi)^2 - S(\pi)^2 = 1$$

Sub. into Thm 8.4.7 again

$$C(x+2\pi) = C(x)C(2\pi) - S(x)S(2\pi) = C(x)$$

$$S(x+2\pi) = S(x)C(2\pi) + C(x)S(2\pi) = S(x)$$

<u>Pf of (xii)</u> By definition of $\pi = 2\gamma$,

$$C(\tfrac{\pi}{2}) = C(\gamma) = 0$$

Cor 8.4.3 $\Rightarrow$ $S(\frac{\pi}{2}) = \pm 1$

Note that $S'(x) = C(x) > 0$, $\forall x \in [0,\gamma) = [0, \frac{\pi}{2})$

$\Rightarrow$ $S(\frac{\pi}{2}) > S(0) = 0$ $\Rightarrow$ $S(\frac{\pi}{2}) = 1$

Then the formulae follow easily from Thm 8.4.7 ※

Ch 9 Infinite Series

§9.1 Absolute Convergence

Recall Eg 3.7.6 (b) Harmonic series

$$\sum_{i=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{is } \underline{\text{divergent}}$$

(since partial sum $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is unbounded)

but Eg 3.7.6 (f) Alternating harmonic series

$$\sum_{i=1}^{n} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{is } \underline{\text{convergent}}$$

$\therefore$ A series $\sum x_n$ may be convergent, but the series $\sum |x_n|$ may be divergent

---

Def 9.1.1

- $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent
- $\sum x_n$ is conditionally convergent (or non-absolutely convergent) if $\sum x_n$ is convergent but $\sum |x_n|$ is divergent.

(i.e. conditionally convergent means convergent but not absolutely convergent)

Eg: Alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

Thm 9.1.2 "Absolutely convergent" $\Rightarrow$ "convergent".

Pf: $\sum |x_n|$ convergent

$\Rightarrow \forall \varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N}$ s.t. (Cauchy Criterion 3.7.4)

if $m > n \geq M(\varepsilon)$, then $|x_{n+1}| + \cdots + |x_m| < \varepsilon$

Let $S_n = x_1 + \cdots + x_n$ be the n^{th} partial sum of $\sum x_n$,

then $\forall m > n \geq M(\varepsilon)$,

$$|S_m - S_n| = |x_{n+1} + \cdots + x_m| \leq |x_{n+1}| + \cdots + |x_m| < \varepsilon.$$

$\therefore \sum x_n$ is convergent. $\#$

Grouping of Series

For a series of $\sum x_n$, one can construct many other series $\sum y_k$ by "grouping the terms": i.e. inserting parentheses that group together finitely many terms, but keeping the order of the terms x_n fixed.

That is

$$y_1 = \sum_{j=1}^{n_1} x_j, \quad y_2 = \sum_{j=n_1+1}^{n_2} x_j, \ldots, y_k = \sum_{j=n_{k-1}+1}^{n_k} x_j, \ldots$$

($n_{k-1} + 1 \leq n_k$, $n_{k-1} = 0$ for $k = 1$)

$\therefore \quad x_1 + x_2 + \cdots + x_n + \cdots$

$= (x_1 + \cdots + x_{n_1}) + (x_{n_1+1} + \cdots x_{n_2}) + (x_{n_2+1} + \cdots) + \cdots$

$= \quad y_1 \quad + \quad y_2 \quad + \quad y_3 \quad + \cdots$

Eg: $1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7}\right) - \frac{1}{8} + \left(\frac{1}{9} - \cdots + \frac{1}{13}\right) - \cdots$

is a grouping the terms of the alternating harmonic series.

(i.e. $y_1 = 1$, $y_2 = -\frac{1}{2}$, $y_3 = \frac{1}{3} - \frac{1}{4}$, $y_4 = \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$

$y_5 = -\frac{1}{8}$, $y_6 = \frac{1}{9} - \cdots + \frac{1}{13}$, $\cdots$)

> Thm 9.1.3 $\sum x_n$ convergent $\Rightarrow$ any series $\sum y_k$ obtained from it by grouping the terms is also convergent, & converges to the same value.

Pf: Let $S_n = n^{th}$ partial sum of $\sum x_n$

$t_k = k^{th}$ partial sum of $\sum y_k$.

If $y_k = \sum_{j=n_{k-1}+1}^{n_k} x_j$,

then $t_1 = y_1 = x_1 + \cdots + x_{n_1} = S_{n_1}$

$t_2 = y_1 + y_2 = \sum_{j=1}^{n_1} x_j + \sum_{j=n_1+1}^{n_2} x_j = x_1 + \cdots + x_{n_2} = S_{n_2}$

$\vdots$

$t_k = S_{n_k}$.

$\therefore$ (t_k) is a subseq. of (S_n)

Since $\sum x_n$ is convergent, $S_n \to S \left(= \sum_{n=1}^{\infty} x_n\right)$ as $n \to \infty$

$\therefore$ $t_k \to S$ as $k \to \infty$

i.e. $\sum y_k$ is convergent and converges to the same value as $\sum x_n$ $\#$

<u>Remark</u>: The converse of Thm 9.1.3 is not true.

Counterexample: Let $\sum x_n = 1 - 1 + 1 - 1 + 1 \cdots$

& $\sum y_k = (1-1) + (1-1) + (1-1) + \cdots$

Then $y_k = 0 \ \forall k \Rightarrow \sum y_k$ is convergent.

But original series $\sum x_n = 1 - 1 + 1 - 1 + 1 \cdots$ is divergent.

<u>Rearrangement of series</u>

(Not grouping any terms, but <u>scrambling</u> the order of the terms.)

> <u>Def 9.1.4</u> $\sum y_k$ is a <u>rearrangement</u> of $\sum x_n$,
>
> if $\exists$ a <u>bijection</u> (ie. one-to-one) $f: \mathbb{N} \to \mathbb{N}$ s.t.
>
> $$y_k = x_{f(k)} \quad \forall k \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$

<u>Remarks</u>: (i) $\sum x_n$ is convergent $\not\Rightarrow$ $\sum y_k$ rearrangement is convergent

(Ex 9.1.3)

(ii) <u>Riemann Thm</u>: If $\sum x_n$ <u>conditionally</u> convergent,

then $\forall c \in \mathbb{R}$, $\exists$ a rearrangement $\sum y_k$ of $\sum x_n$ such that

$$\sum_{k=1}^{\infty} y_k = c \qquad \text{(Pf omitted)}$$

Thm 9.1.5 If $\sum x_n$ is <u>absolutely</u> convergent, then <u>any rearrangement</u> $\sum y_k$ of $\sum x_n$ converges to the same value.

<u>Pf</u>: $\sum x_n$ absolutely convergent $\Rightarrow$ $\sum x_n$ convergent.

Let $x = \sum_{n=1}^{\infty} x_n$, and $s_n = \sum_{k=1}^{n} x_k$.

Then $s_n \to x$ as $n \to \infty$

$\therefore$ $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t.

if $n \geq N_1$, $|s_n - x| < \varepsilon$.

On the other hand, $\sum |x_n|$ convergent

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N_2 \in \mathbb{N}$ s.t.

if $q > l \geq N_2$, then $|x_{l+1}| + |x_{l+2}| + \cdots + |x_q| < \varepsilon$

Therefore, for $N = \max\{N_1, N_2\}$,

if $n, q > N$, $\quad |s_n - x| < \varepsilon$ and $\qquad$ —(*)

$|x_{N+1}| + |x_{N+2}| + \cdots + |x_q| < \varepsilon$

Let $\sum y_k$ be a rearrangement of $\sum x_n$ given by the bijection $f: \mathbb{N} \to \mathbb{N}$, i.e. $y_k = x_{f(k)}$, $\forall k \in \mathbb{N}$.

Let $M = \max\{f^{-1}(1), \cdots, f^{-1}(N)\}$,

then all the terms $x_1, \cdots, x_N$ are contained in

$\{y_1, \cdots, y_M\}$.

$\therefore$ If $t_m = \sum_{k=1}^{m} y_k$, then $\forall m \geq M$, ($\&$ $n > N$)

$$t_m - s_n = (y_1 + \cdots + y_M + \cdots + y_m) - (x_1 + \cdots + x_N + \cdots + x_n)$$

$$= \underbrace{(y_1 + \cdots + y_M) - (x_1 + \cdots + x_N)}_{(\text{no } x_1, \cdots, x_N \text{ remain})} + \underbrace{(y_{M+1} + \cdots + y_m) - (x_{N+1} + \cdots + x_n)}_{(\text{no } x_1, \cdots, x_N \text{ in these terms})}$$

is a sum of finite number of terms x_k with $k > N$.

$$\Rightarrow \quad |t_m - s_n| \leq \sum_{k=N+1}^{q} |x_k| \quad \text{for some } q$$

By $(*)$, $\quad |t_m - s_n| < \varepsilon$.

Hence, $\forall \varepsilon > 0$, $\exists M > 0$ such that

if $m \geq M$, $\quad |t_m - x| \leq |t_m - s_n| + |s_n - x| < \varepsilon + \varepsilon = 2\varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $\lim_{m \to \infty} t_m = x$

$\therefore \quad \sum y_k \to x = \sum x_n$.

$\#$