(Contd from last time)

Let $c \in I$, then mean value thm $\Rightarrow$ for $x \in I$ & $x \neq c$

$$(f_m - f_n)(x) - (f_m - f_n)(c) = (f_m' - f_n')(z)\ (x - c)$$

for some z between x & c.

$$\therefore \quad \left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| = |f_m'(z) - f_n'(z)|$$

$$\leq \|f_m' - f_n'\|_I$$

Hence $\forall \varepsilon > 0$,

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| < \frac{\varepsilon}{2(b-a)} \quad \text{for } m, n \geq H_1$$

Letting $m \to \infty$ and using $f_m \rightrightarrows f$, we have for $x \neq c$

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \frac{\varepsilon}{2(b-a)} \quad \text{for } n \geq H_1$$

Now using $f_n' \rightrightarrows g$ again

for the same $\varepsilon > 0$, $\exists\, N = N(\varepsilon) \in \mathbb{N}$ s.t.

$$|f_n'(c) - g(c)| < \varepsilon \quad \text{for } n \geq N$$

Then let $K = \max\{H_1, N\} \in \mathbb{N}$

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_K(x) - f_K(c)}{x - c} \right|$$

$$+ \left| \frac{f_K(x) - f_K(c)}{x - c} - f_K'(c) \right| + |f_K'(c) - g(c)|$$

$$< \left(1+\frac{1}{2(b-a)}\right)\varepsilon + \left|\frac{f_K(x)-f_K(c)}{x-c} - f_K'(c)\right|$$

Note that for the same $\varepsilon>0$, $\exists\ \delta_{\varepsilon,c}>0$ such that

$$\left|\frac{f_K(x)-f_K(c)}{x-c} - f_K'(c)\right| < \varepsilon, \quad \text{if } |x-c|<\delta_{\varepsilon,c} \ (x\neq c).$$

Therefore, we have proved that $\forall\, \varepsilon>0$, $\exists\ \delta_{\varepsilon,c}>0$

s.t. $\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| < \left(2+\frac{1}{2(b-a)}\right)\varepsilon$ provide $|x-c|<\delta_{\varepsilon,c}$.

Since $\varepsilon>0$ is arbitrary,

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ exists \& equals } g(c).$$

As $c\in I$ is arbitrary, f is differentiable on I and $f'=g$.

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Interchange of Limit and Integral

Thm 8.2.4 Let $\left\{ \begin{array}{l} \bullet\ f_n \in \mathcal{R}[a,b] \text{ for } n = 1,2,3,\cdots \quad \text{(Riemann integrable)} \\ \bullet\ f_n \rightrightarrows f \text{ on } [a,b] \text{ (converges uniformly on } [a,b] \text{ to } f) \end{array} \right.$

Then $f \in \mathcal{R}[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

(i.e. f_n converges <u>uniformly</u> $\Rightarrow \lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$)

Pf: By Cauchy Criterion for Uniform Convergence (Thm 8.1.10),

$\forall \varepsilon > 0, \ \exists H(\varepsilon) > 0$ s.t.

if $m > n \geq H(\varepsilon)$, then $\| f_m - f_n \|_{[a,b]} < \varepsilon$

i.e. $-\varepsilon < f_m(x) - f_n(x) < \varepsilon \quad \forall x \in [a,b]$

Hence
$$-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a) \quad \text{---} (*)_1$$

i.e.
$$\left| \int_a^b f_m - \int_a^b f_n \right| \leq \varepsilon(b-a)$$

Since $\varepsilon > 0$ is arbitrary, this implies the seq. of numbers $\left(\int_a^b f_n\right)$ is a Cauchy sequence.

$\therefore \quad \lim_{n\to\infty} \int_a^b f_n = A$ exists, (denoted by A).

$\Rightarrow \forall \varepsilon > 0, \exists K(\varepsilon) > 0$ s.t. $\left| \int_a^b f_n - A \right| < \varepsilon$, for $n \geq K(\varepsilon)$. $\text{---} (*)_2$

And letting $m \to \infty$ in the ineq. before $(*)_1$, we have

$\forall \varepsilon > 0$, $\exists\, H(\varepsilon) > 0$ s.t. if $n \geq H(\varepsilon)$, then

$$-\varepsilon \leq f(x) - f_n(x) \leq \varepsilon$$

i.e. $\quad |f_n(x) - f(x)| \leq \varepsilon \qquad \forall\, x \in [a,b] \quad —— (*)_3$

Now, let $\dot{\mathcal{P}} = \{ [x_{i-1}, x_i], t_i \}_{i=1}^{\ell}$ be a tagged partition of $[a,b]$.

If $n \geq \max\{H(\varepsilon), K(\varepsilon)\}$, we have

$$\begin{aligned}
|S(f_n; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})| &= \left| \sum_{i=1}^{\ell} f_n(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{\ell} f(t_i)(x_i - x_{i-1}) \right| \\
&= \left| \sum_{i=1}^{\ell} (f_n(t_i) - f(t_i))(x_i - x_{i-1}) \right| \\
&\leq \sum_{i=1}^{\ell} |f_n(t_i) - f(t_i)| (x_i - x_{i-1}) \\
&\leq \varepsilon \sum_{i=1}^{\ell} (x_i - x_{i-1}) \qquad (\text{by } (*)_3) \\
&= \varepsilon (b-a)
\end{aligned}$$

Then

$$\begin{aligned}
|S(f; \dot{\mathcal{P}}) - A| &\leq |S(f; \dot{\mathcal{P}}) - S(f_n; \dot{\mathcal{P}})| + |S(f_n, \dot{\mathcal{P}}) - A| \\
&\leq \varepsilon(b-a) + \left|S(f_n; \dot{\mathcal{P}}) - \int_a^b f_n\right| + \left|\int_a^b f_n - A\right|. \\
&< \varepsilon(b-a+1) + \left|S(f_n; \dot{\mathcal{P}}) - \int_a^b f_n\right|
\end{aligned}$$

Finally, fix an $n_0 \geq \max\{H(\varepsilon), K(\varepsilon)\}$ and using $f_{n_0} \in \mathcal{R}[a,b]$, $\exists\, \delta_{\varepsilon, n_0} > 0$ (depends on n_0 too) s.t.

if $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon, n_0}$, then $|S(f_{n_0}; \dot{\mathcal{P}}) - \int_a^b f_{n_0}| < \varepsilon$.

Hence $\forall \varepsilon > 0$, if $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon, n_0}$, we have

$$|S(f; \dot{\mathcal{P}}) - A| < \varepsilon(b-a+1) + \varepsilon = \varepsilon(b-a+2).$$

Since $\varepsilon > 0$ is arbitrary, we have proved that

$$f \in R[a,b] \text{ and } \int_a^b f = A = \lim_{n\to\infty} \int_a^b f_n.$$

$\divideontimes$

Thm 8.2.5 ((Uniform) Bounded Convergence Theorem)

Let
- $f_n \in R[a,b]$ $\forall n = 1, 2, 3, \cdots$ (Riemann integrable)
- $f_n \to f$ on $[a,b]$ (pointwise convergence)
- $f \in R[a,b$
- $\exists B > 0$ such that $\|f_n\|_{[a,b]} \le B$, $\forall n = 1, 2, 3 \cdots$
  (i.e. $|f_n(x)| \le B$, $\forall x \in [a,b]$ & $\forall n = 1, 2, 3, \cdots$)

Then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f = \int_a^b \lim_{n\to\infty} f_n$

Pf: Omitted

Remark: The condition in Bounded Convergence Thm is weaker than Thm 8.2.4