Ch 8 Sequences of Functions

§8.1 Pointwise and Uniform Convergence

Def: Let $A \subseteq \mathbb{R}$ be a set.

If $\forall n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, there is a function

$$f_n : A \to \mathbb{R}$$

Then (f_n) is called a sequence of functions on A (to $\mathbb{R}$).

Remark: If (f_n) is a seq. of functions on A, then $\forall x \in A$, $(f_n(x))$ is a sequence of numbers in $\mathbb{R}$.

Def 8.1.1 (Pointwise Convergence)

Let
- (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$,
- $f : A_0 \to \mathbb{R}$, where $A_0 \subseteq A$

We say that the sequence (f_n) converges on A_0 to f if

$$\lim_{n \to \infty} f_n(x) = f(x), \quad \forall x \in A_0.$$

In this case,
- f is called the limit on A_0 of the sequence (f_n).
- (f_n) is said to be convergent on A_0, or (f_n) converges pointwise on A_0.

Remarks (i) Usually, we choose

$$A_0 = \{ x \in A : (f_n(x)) \text{ converges} \}$$

(ii) Symbols:

$$\begin{cases} \bullet\ f = \lim f_n \quad \text{on } A_0, \text{ or} & \left(\begin{array}{c} f = \lim (f_n) \\ \text{in the Textbook} \end{array} \right) \\ \bullet\ f_n \to f \quad \text{on } A_0 \end{cases}$$

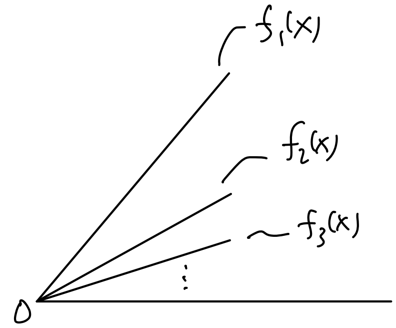
or

$$\begin{cases} \bullet\ f(x) = \lim f_n(x) \quad \text{for } x \in A_0, \text{ or} \\ \bullet\ f_n(x) \to f(x) \quad \text{for } x \in A_0 \end{cases}$$

Eg 8.1.2

(a) $f_n(x) = \frac{x}{n}$, $\forall x \in \mathbb{R}$;

$f(x) = 0$, $\forall x \in \mathbb{R}$

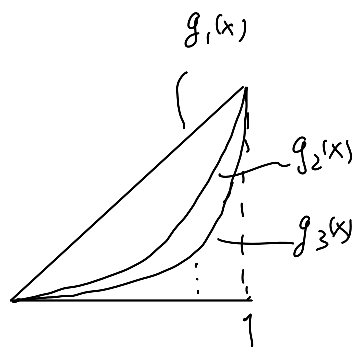


Then $\lim_{n\to\infty} \frac{x}{n} = 0$, $\forall x \in \mathbb{R}$, implies

$$\lim f_n = f \qquad (A_0 = \mathbb{R})$$

(i.e. $\lim_{n\to\infty} \frac{x}{n} = 0$)

(b) $g_n(x) = x^n$, $\forall x \in \mathbb{R}$ $(n = 1, 2, 3, \cdots)$



If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$

If $|x| > 1$, then x^n diverges.

and $1^n = 1$, $\forall n = 1, 2, 3 \cdots$

$(-1)^n$ diverges

$\left(\therefore \ A_0 = \{ x \in \mathbb{R} : -1 < x \leq 1 \} \right)$

and $\quad x^n \longrightarrow g(x) = \begin{cases} 0 , & -1 < x < 1 \\ 1 , & x = 1 \end{cases} \quad$ on $(-1, 1]$

(↑ discontinuous at $x=1$)

(c) Let $h_n(x) = \dfrac{x^2 + nx}{n}$, $\forall x \in \mathbb{R}$ and $\quad$ (see Textbook for graphs)

$h(x) = x \quad , \forall x \in \mathbb{R}$

Then $\forall x \in \mathbb{R}$, $\displaystyle \lim_{n \to \infty} h_n(x) = \lim_{n \to \infty} \left(\frac{x^2}{n} + x \right) = x = h(x)$

$(\therefore A_0 = \mathbb{R})$

(d) $F_n(x) = \frac{1}{n} \sin(n(x+1))$, $\forall x \in \mathbb{R}$, and $\quad$ (see Textbook for graphs)

$F(x) = 0 \quad , \forall x \in \mathbb{R}$

Since $\forall x \in \mathbb{R}$

$$| F_n(x) - F(x) | = \frac{1}{n} | \sin(n(x+1)) | \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

$\therefore \ F_n \to F$ on $\mathbb{R}$ $\quad$ (i.e. $A_0 = \mathbb{R}$)

Lemma 8.1.3 A seq. $f_n : A \to \mathbb{R}$ <u>converges</u> to $f : A_0 \to \mathbb{R}$ $(A_0 \subseteq A)$ if and only if $\forall \varepsilon > 0$ and $\forall x \in A_0$, $\exists K(\varepsilon, x) \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq K(\varepsilon, x)$.

Pf: This follows from the definition of "limit of sequences". Note that for different $x \in A_0$, $(f_n(x))$ are different sequences and hence the number K in the def of "limit of seq" depends on x too. Therefore, we have a natural number $K(\varepsilon, x)$ (not just $K(\varepsilon)$ in general).

eg 8.1.2(b) For $|x| < 1$, $\quad |g_n(x) - g(x)| = |x^n| = |x|^n < \varepsilon$

Suppose $\varepsilon < 1$, then $\quad n \log|x| < \log \varepsilon$

(note both $\log \varepsilon, \log|x| < 0$) $\Rightarrow \quad n > \dfrac{\log 1/\varepsilon}{\log 1/|x|}$

$\therefore$ One need to choose $K(\varepsilon, x) = \left[\dfrac{\log 1/\varepsilon}{\log 1/|x|}\right] + 1$

(largest integer $\leq \dfrac{\log 1/\varepsilon}{\log 1/|x|}$, i.e. integer part of the number)

which <u>depends on x</u>, and $K(\varepsilon, x) \to +\infty$ as $|x| \to 1$.

$\therefore$ Can't choose $K(\varepsilon)$ that works $\forall x \in (-1, 1]$