§7.3 The Fundamental Theorem

Recall: A function $F:[a,b]\to\mathbb{R}$ is called an antiderivative

or a primitive of $f:[a,b]\to\mathbb{R}$ on $[a,b]$ if

$$F'(x)=f(x),\ \forall\, x\in[a,b]$$

(One sided derivatives at $x=a$ & $x=b$)

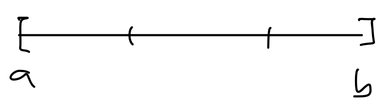
Thm 7.3.1 (Fundamental Theorem of Calculus (1st Form))

Suppose $\begin{cases} \bullet\ f, F:[a,b]\to\mathbb{R} \text{ functions}, \\ \bullet\ E = \text{finite set of } [a,b] \end{cases}$ (E for exceptional set)

such that $\begin{cases} \text{(a) } F \text{ is continuous on } [a,b], \\ \text{(b) } F'(x)=f(x)\ \ \forall\, x\in[a,b]\setminus E, \\ \text{(c) } f\in\mathcal{R}[a,b] \end{cases}$

Then $$\boxed{\int_a^b f = F(b)-F(a)}$$

Pf: With the finite # of points in E,



$[a,b]$ is subdivided into finite number of subintervals such that $F'(x)=f(x)$ on the subintervals except possibly at endpoints. Then by Thm 7.1.3 & Thm 7.2.9, one can reduce the proof of the

Thm to the case that $E=\{a,b\}$ two end points only

i.e. $F'(x)=f(x),\ \forall x\in(a,b)$.

(Exercise 7.3.1 of the Textbook)

For this special case, consider any $\varepsilon>0$.

Then $f\in\mathcal{R}[a,b]$ (assumption (c)) $\Rightarrow$

$\exists\ \delta_\varepsilon>0$ such that

if $\dot{\mathcal{P}}=\{[x_{i-1},x_i],t_i\}_{i=1}^{n}$ satisfies $\|\dot{\mathcal{P}}\|<\delta_\varepsilon$, (any tags t_i)

then $$\left|S(f,\dot{\mathcal{P}})-\int_a^b f\right|<\varepsilon. \qquad (*)$$

By Mean Value Thm 6.2.4, $\exists\ u_i\in(x_{i-1},x_i)$ s.t.

$$\begin{aligned}F(x_i)-F(x_{i-1}) &= F'(u_i)(x_i-x_{i-1})\\ &= f(u_i)(x_i-x_{i-1}),\quad \forall i=1,\dots,n\end{aligned}$$

since $F'=f$ exists on (a,b) (assumption (b) of the special case)

Hence $$\begin{aligned}F(b)-F(a) &= \sum_{i=1}^{n}\left[F(x_i)-F(x_{i-1})\right]\\ &= \sum_{i=1}^{n} f(u_i)(x_i-x_{i-1})\end{aligned}$$

Define the tagged partition $\ddot{\mathcal{P}}_u=\{[x_{i-1},x_i],u_i\}_{i=1}^{n}$

(same partition with new tags).

Then $\|\dot{\mathcal{P}}_u\| < \delta_\varepsilon$ and

$$F(b)-F(a) = S(f, \dot{\mathcal{P}}_u)$$

$$\therefore \quad \left| F(b)-F(a) - \int_a^b f \right| < \varepsilon, \quad \text{by } (*)$$

Since $\varepsilon>0$ is arbitrary, $\int_a^b f = F(b)-F(a)$. #

<u>Remarks</u>: (i) If $E=\emptyset$, then assumption (b) $\Rightarrow$ assumption (a).

(ii) One may allow f defined on $[a,b]$ except finite number of points as one can extend f to all $x\in[a,b]$ by setting $f(c)=0$ for $c\notin \text{domain}(f)$ originally.

(iii) F differentiable on $[a,b] \not\Rightarrow F' \in \mathcal{R}[a,b]$

$\therefore$ assumption (c) is not automatically satisfied even $E=\emptyset$ & assumption (b) is satisfied. (Eg 7.3.2 (e))

<u>Eg 7.3.2</u>

(a) • $F(x)=\frac{1}{2}x^2$, $\forall x\in[a,b]$ is continuous on $[a,b]$,

• $F'(x)=x$, $\forall x\in[a,b]$ ($\therefore E=\emptyset$)

• $F'(x)=x\in\mathcal{R}[a,b]$ (says by Thm 7.2.7, cts $\Rightarrow$ integrable)

$$\therefore \quad \int_a^b x\,dx = F(b)-F(a) = \frac{1}{2}(b^2-a^2).$$

(b) Suppose $[a,b]$ is a closed interval s.t. $\left(\text{Arctan}\, x = \tan^{-1} x\right)$

$G(x) = \text{Arctan}\, x$ is defined on $[a,b]$ $\left(\text{for instance } [a,b] \subset (-\frac{\pi}{2}, \frac{\pi}{2})\right)$

Then $G'(x) = \dfrac{1}{x^2+1}$, $\forall x \in [a,b]$ & is <u>continuous</u> on $[a,b]$

$\therefore$ (b) satisfied with $E = \emptyset$. (with $f(x) = \frac{1}{x^2+1}$)

Hence (a) satisfied automatically.

And Thm 7.2.7 $\Rightarrow$ (c) is also satisfied.

$$\therefore \int_a^b \frac{dx}{x^2+1} = \text{Arctan}\, b - \text{Arctan}\, a.$$

(c) $A(x) = |x|$ for $x \in [-10, 10]$. cts. $\left(\begin{array}{l}\text{one can do } [-\alpha, \beta] \\ \text{for any } \alpha, \beta > 0\end{array}\right)$

Then
$$A'(x) = \begin{cases} 1, & \text{for } x \in (0, 10] \\ \text{doesn't exist}, & \text{for } x = 0 \\ -1, & \text{for } x \in [-10, 0) \end{cases}$$

Recall the signum function
$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$\therefore$ $A'(x) = \text{sgn}(x)$ $\forall x \in [-10, 10] \setminus \{0\}$ $(E = \{0\})$

Note that $\text{sgn}(x)$ is a step function, Thm 7.2.5

$\Rightarrow$ $\text{sgn}(x) \in \mathcal{R}[-10, 10]$. with one degenerated interval

Hence $$\int_{-10}^{10} \text{sgn}(x)\, dx = A(10) - A(-10) = 10 - 10 = 0.$$

(d) $H(x) = 2\sqrt{x}$ on $[0,b]$.

Then $H(x)$ cts on $[0,b]$,

$$H'(x) = \frac{1}{\sqrt{x}} \quad \forall\, x \in (0,b] \qquad (E = \{0\})$$

Note that $h(x) = \frac{1}{\sqrt{x}}$ is unbounded on $[0,b]$,

$h \notin R[0,b]$ (No matter how we define $H'(0)$)

$\therefore$ Fundamental Thm 7.3.1 <u>doesn't</u> apply!

(Need to consider improper integrals, which is equivalent to applying Thm 7.3.1 to $[\varepsilon, b]$, and then letting $\varepsilon \to 0$.)

(e)

$$K(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right), & x \in (0,1] \\ 0, & x = 0 \end{cases}$$

Then

$$K'(x) = \begin{cases} 2x \cos\frac{1}{x^2} + \frac{2}{x}\sin\left(\frac{1}{x^2}\right), & x \in (0,1] \\ 0, & \text{if } x = 0 \quad (\text{eg } 6.1.7(c)) \end{cases}$$

That is, K differentiable on $[0,1]$, & hence cts on $[0,1]$.

However K' is unbounded and

therefore $K' \notin R[0,1]$, assumption (c) doesn't satisfy!

Def 7.3.3 : If $f \in \mathcal{R}[a,b]$, then the function defined by

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b],$$

is called the indefinite integral of f with basepoint a.

(One may use other point as base point & is still called indefinite integral (Ex 7.3.6))

Thm 7.3.4 If $f \in \mathcal{R}[a,b]$, then

$$F(z) = \int_a^z f \quad \text{is continuous on } [a,b]$$

and in fact, if $|f(x)| \le M$, $\forall x \in [a,b]$, then

$$(*) \qquad |F(z) - F(w)| \le M|z-w|, \quad \forall z,w \in [a,b].$$

Remarks: (i) M exists because $f \in \mathcal{R}[a,b] \Rightarrow f$ is bdd

(ii) $(*)$ is called a Lipschitz condition, much stronger than just continuity.

Pf $\forall z,w \in [a,b]$ with $w \le z$, Additivity Thm 7.2.9 $\Rightarrow$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f$$

$$\therefore \quad F(z) - F(w) = \int_w^z f.$$

If $\qquad -M \le f(x) \le M, \quad \forall x \in [a,b],$

Thm 7.1.5 (c) $\Rightarrow \quad -M(z-w) \le \int_w^z f \le M(z-w)$

$\therefore \quad |F(z)-F(w)| = \left|\int_w^z f\right| \le M(z-w) = M|z-w|$

(since $w \le z$)

Clearly, the case $z \le w$ follows immediately too. #

Thm 7.3.5 (Fundamental Theorem of Calculus (2nd Form))

Let $f \in \mathcal{R}[a,b]$ and continuous at c.

Then $F(z) = \int_a^z f$ is differentiable at $z=c$ and

$$F'(c) = f(c).$$

Pf We'll prove only for the right-hand derivative

$$\lim_{h\to 0+} \frac{F(c+h)-F(c)}{h} = f(c)$$

The left-hand derivative can be handled similarly.

Therefore, we assume $c \in [a,b)$.

Since f is continuous at c, $\forall \varepsilon > 0, \exists \eta_\varepsilon > 0$ s.t. if

(*) $\quad |f(x)-f(c)| < \varepsilon, \quad \forall x \in [c, c+\eta_\varepsilon)$. (consider only right side)

Let $h \in (0, \eta_\varepsilon)$, then Additivity Thm 7.2.9 (Cor 7.2.10)

$\Rightarrow$ $f \in \mathcal{R}[a, c+h]$, $\mathcal{R}[a,c]$ & $\mathcal{R}[c, c+h]$ and

$$\int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f$$

i.e. $$F(c+h) - F(c) = \int_c^{c+h} f$$

By $(*)$ $\quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \ \forall x \in [c, c+\eta_\varepsilon)$

we have

$$(f(c) - \varepsilon) h \leq \int_c^{c+h} f \leq (f(c) + \varepsilon) h,$$

which implies

$$f(c) - \varepsilon \leq \frac{F(c+h) - F(c)}{h} \leq f(c) + \varepsilon$$

$$\Rightarrow \quad \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \leq \varepsilon, \quad \forall h \in (0, \eta_\varepsilon)$$

It proves that $\displaystyle \lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$

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Thm 7.3.6 If f is continuous on $[a,b]$, then

- $F(x) = \int_a^x f$ is differentiable on $[a,b]$, and
- $F'(x) = f(x)$, $\forall x \in [a,b]$

Pf: f cts on $[a,b]$ $\Rightarrow$ $f \in \mathcal{R}[a,b]$ & cts at every pt. $c \in [a,b]$

#

Eg 7.3.7

(a) $f(x) = \text{sgn}\, x$ on $[-1,1]$.

Then • $f \in \mathcal{R}[-1,1]$ (step function with a degenerated interval)

• f not continuous at $x=0$, but continuous $\forall x \in [-1,1] \setminus \{0\}$.

Simply calculation: indefinite integral with basepoint -1 is

$$F(x) = \int_{-1}^{x} \text{sgn}(x)\,dx = |x| - 1 \quad (\text{Ex!})$$

One can see that $F'(0)$ doesn't exist ("f cts at c" is a necessary condition)

and F' is <u>not</u> an antiderivative of $f(x) = \text{sgn}(x)$.

(b) Let h = Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \ \& \ \underset{\neq 0}{m}, n \text{ have no common factors } (\gcd(m,n)=1), \ n \in \mathbb{N} = \{1,2,3,\cdots\} \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational } \& \ x \in [0,1]. \end{cases}$$

Then by Eg 7.1.7, one concludes that

$$H(x) = \int_0^x h \equiv 0, \quad \forall x \in [0,1]$$

$\Rightarrow$ $H'(x) = 0$ exists $\forall x \in [0,1]$

However, $H'(x) \neq h(x)$, $\forall$ rational $x \in [0,1]$.

Thm 7.3.8 (<u>Substitution Theorem</u>)

Let
- $f : I \to \mathbb{R}$ <u>cts</u>, $\quad (I = \text{interval})$
- $\varphi : [\alpha, \beta] \to \mathbb{R}$ st. $\varphi'(t)$ <u>exists & cts</u> $\forall t \in [\alpha, \beta]$,
  (i.e. φ has a <u>continuous derivative</u>)
- $\varphi([\alpha,\beta]) \subset I$ $\quad$ $\left(\begin{tikzcd} {[\alpha,\beta]} \arrow[r, "\varphi"] \arrow[rr, bend right, "f\circ\varphi"'] & I \arrow[r, "f"] & \mathbb{R} \end{tikzcd} \right)$

Then
$$\int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

Notes: (i) t & x in the formula are dummy variables, just using them for convenient in practice:

thinking of change of variables $x = \varphi(t)$

In fact, the formula can be written as
$$\int_{\alpha}^{\beta} (f\circ\varphi)\cdot\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f$$

(ii) The formula holds for $\varphi(\beta) \le \varphi(\alpha)$ as we defined before.

<u>Pf of Thm 7.3.8</u>: Ex 7.3.17 (Easy application of Fundamental Thm & Chain rule)

<u>Eg 7.3.9</u> <u>Omitted</u>

Lebesgue's Integrability Criterion

Def 7.3.10

(a) A set $Z \subset \mathbb{R}$ is said to be a null set (set of measure zero)

if $\forall \varepsilon > 0$, $\exists$ a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals (could be overlapped) such that

$$Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} \underbrace{(b_k - a_k)}_{\text{length of interval } (a_k, b_k)} \leq \varepsilon$$

(b) If $Q(x)$ is a statement about $x \in I$, we say that

" $Q(x)$ holds almost everywhere on I "

(or " $Q(x)$ holds for almost every (almost all) $x \in I$")

if $\exists$ a null set $Z \subset I$ s.t.

$$Q(x) \text{ holds } \forall x \in I \setminus Z.$$

In this case, we write

$$Q(x) \text{ for a.e. } x \in I.$$

Remarks : (i) "null set" may means "empty set" for some people. So "set of measure zero" is used more often.

(ii) Def (a) means Z can be covered by a set of arbitrary small total length. (Kind of "length of $Z = 0$", but it is difficult to define "length" of arbitrary sets in $\mathbb{R}$.)

Eg 7.3.11 $\mathbb{Q}_1$ = set of rational numbers in $[0,1]$ is a null set.

(set of measure zero)

Pf: $\mathbb{Q}_1$ is countable and can be written as

$$\mathbb{Q}_1 = \{r_1, r_2, r_3, \cdots\}$$

Given $\varepsilon > 0$, define open intervals

$$J_k = \left(r_k - \frac{\varepsilon}{2^{k+1}}, r_k + \frac{\varepsilon}{2^{k+1}}\right), \quad k = 1, 2, \cdots$$

Clearly $r_k \in J_k$ and length of $J_k = \frac{\varepsilon}{2^k}$.

$$\therefore \mathbb{Q}_1 \subset \bigcup_{k=1}^{\infty} J_k \text{ and } \sum_{k=1}^{\infty} \text{length of } J_k = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mathbb{Q}_1$ is a null set.

(From the proof, it is clear that it doesn't use the fact that r_k are rational. Hence, the proof can be used to prove that:

$\boxed{\text{Every countable set is a null set (set of measure zero)}}$

("countable infinite" can be proved similarly, "countable finite" are included by dropping the tail of the infinite series)

Thm 7.3.12 (Lebesgue's Integrability Criterion)

A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a,b]$

(Pf: Omitted. See App. C of the Textbook)

Eg 7.3.13

(a) Every step function on $[a,b]$ is bdd & has a finite set of points of discontinuity which is a set of measure zero and hence every step function on $[a,b]$ is Riemann integrable.

(b) Every monotone function on $[a,b]$ is Riemann integrable

In fact, monotone functions are bounded &

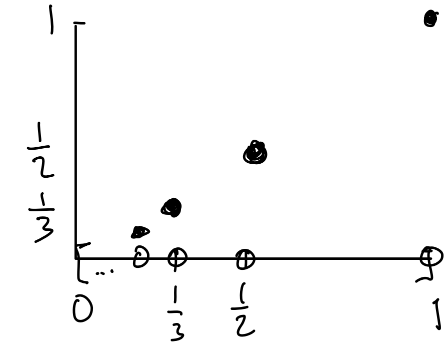
Thm 5.6.4 $\Rightarrow$ set of points of discontinuity of a monotonic function is countable.

Hence, it is a null set.

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$ it is Riemann integrable

(c) (eg 7.1.4(d))

$$G(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \quad (n = 1, 2, \cdots) \\ 0, & \text{elsewhere on } [0,1] \end{cases}$$



is bounded, and

Set of discontinuity $= \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

is countable hence measure zero.

Lebesgue's Integrability criterion $\Rightarrow$ $G(x)$ is Riemann integrable

(d) (Eg 7.2.2 (b), not integrable)

Dirichlet function $f(x) = \begin{cases} 1, & \text{if } x \text{ rational}, x \in [0,1] \\ 0, & \text{if } x \text{ irrational}, x \in [0,1]. \end{cases}$

is bounded.

set of discontinuity $= [0,1]$ (discontinuous at every $x \in [0,1]$)

which can be shown that it is <u>not a null set</u> (Omitted)

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$

Dirichlet function is not Riemann integrable.

(e) (eg 7.1.7) Thomae's function

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in [0,1] \ \& \ m, n \text{ have no common factors} \\ & \quad (m \neq 0,\ n \in \mathbb{N} = \{1,2,3,\cdots\}) \quad (\gcd(m,n)=1) \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational } \& \ x \in [0,1]. \end{cases}$$

is bounded & (by eg. 5.1.6(h))

set of discontinuity $= \mathbb{Q}_1$ (set of rational numbers in $[0,1]$)

which is of measure zero (eg 7.3.11)

$\therefore$ Lebesgue's Integrability criterion $\Rightarrow$

Thomae's function is Riemann integrable on $[0,1]$.