Thm 7.2.9 (Additivity Theorem)

Let $f : [a,b] \to \mathbb{R}$ & $c \in (a,b)$. $(a<b)$

Then $f \in \mathcal{R}[a,b] \iff f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f|_{[c,b]} \in \mathcal{R}[c,b]$.

In this case $\int_a^b f = \int_a^c f + \int_c^b f$

Pf ($\Rightarrow$) By Cauchy Criterion (Thm 7.2.1)

$f \in \mathcal{R}[a,b]$

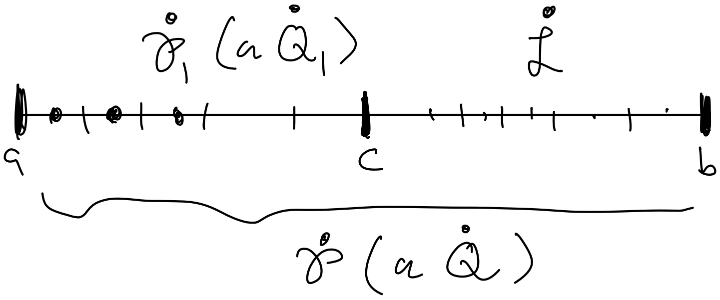
$\iff \forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. $\forall \dot{\mathcal{P}}, \dot{\mathcal{Q}}$ with $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$

we have $|S(f, \dot{\mathcal{P}}) - S(f, \dot{\mathcal{Q}})| < \varepsilon$. —— $(*)_1$

Now we want to show that the same $\eta_\varepsilon > 0$ works for the restriction $f_1 = f_{[a,c]} : [a,c] \to \mathbb{R}$.

Suppose $\dot{\mathcal{P}}_1$ & $\dot{\mathcal{Q}}_1$ be two ~~tagged~~ partitions of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \eta_\varepsilon$ & $\|\dot{\mathcal{Q}}_1\| < \eta_\varepsilon$.

Define now tagged partitions $\dot{\mathcal{P}}$ & $\dot{\mathcal{Q}}$ of $[a,b]$ by adding a tagged partition $\dot{\mathcal{L}}$ of $[c,b]$ with $\|\dot{\mathcal{L}}\| < \eta_\varepsilon$ to $\dot{\mathcal{P}}_1$ & $\dot{\mathcal{Q}}_1$



Then clearly $\|\mathring{\mathcal{P}}\| < \eta_\varepsilon$ & $\|\mathring{Q}\| < \eta_\varepsilon$

By $(*)_1$,

$$|S(f, \mathring{\mathcal{P}}) - S(f, \mathring{Q})| < \varepsilon.$$

On the other hand

$$S(f, \mathring{\mathcal{P}}) = \underbrace{\sum_{x_i \le c} f(t_i)(x_i - x_{i-1})}_{\mathring{\mathcal{P}}_1} + \underbrace{\sum_{x_{i-1} \ge c} f(t_i)(x_i - x_{i-1})}_{\mathring{\mathcal{L}}}$$

and
$$S(f, \mathring{Q}) = \underbrace{\sum_{x_i' \le c} f(t_i')(x_i' - x_{i-1}')}_{\mathring{Q}_1} + \underbrace{\sum_{x_{i-1} \ge c} f(t_i)(x_i - x_{i-1})}_{\mathring{\mathcal{L}}}$$

$$\therefore \quad S(f, \mathring{\mathcal{P}}) - S(f, \mathring{Q}) = S(f_1, \mathring{\mathcal{P}}_1) - S(f_1, \mathring{Q}_1)$$

$$\Rightarrow \qquad |S(f_1, \mathring{\mathcal{P}}_1) - S(f_1, \mathring{Q}_1)| < \varepsilon$$

Hence $f_1 : [a, c] \to \mathbb{R}$ satisfies Cauchy Criterion.

Therefore $f_1 \in \mathcal{R}[a, c]$.

Similarly, we have $\quad f_2 = f|_{[c,b]} \in \mathcal{R}[c, b]$.

($\Leftarrow$) Suppose $f_1 = f|_{[a,c]} \in \mathcal{R}[a,c]$ & $f_2 = f|_{[c,b]} \in \mathcal{R}[c,b]$.

Then Boundedness Thm 7.1.6 $\Rightarrow$ $f|_{[a,c]}$ & $f|_{[c,b]}$ are bdd.

$\Rightarrow$ f is bounded on $[a,b]$.

i.e. $\exists M > 0$ such that $|f(x)| \le M$, $\forall x \in [a,b]$.

Next let $L_1 = \int_a^c f_1 \left(= \int_a^c f\right)$ &

$L_2 = \int_c^b f_2 \left(= \int_c^b f\right)$

Then $\forall \varepsilon > 0$,

$\exists \delta' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_1$ of $[a,c]$ with $\|\dot{\mathcal{P}}_1\| < \delta'$,

we have $|S(f_1, \dot{\mathcal{P}}_1) - L_1| < \varepsilon/3$

and

$\exists \delta'' > 0$ s.t. $\forall$ tagged partition $\dot{\mathcal{P}}_2$ of $[c,b]$ with $\|\dot{\mathcal{P}}_2\| < \delta''$,

we have $|S(f_2, \dot{\mathcal{P}}_2) - L_2| < \varepsilon/3$.

Now let $\delta_\varepsilon = \min\{\delta', \delta'', \frac{\varepsilon}{6M}\} > 0$ &

<u>Claim</u>: If $\dot{Q}$ is a tagged partition of $[a,b]$ with $\|\dot{Q}\| < \delta_\varepsilon$, then

$$|S(f; \dot{Q}) - (L_1 + L_2)| < \varepsilon.$$

If the Claim holds, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = L_1 + L_2$ and we're done.

<u>Pf of claim</u>

Let $\dot{Q} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^n$

then $x_i - x_{i-1} < \delta_\varepsilon$, $\forall i = 1, \cdots, n$.

Case (i) $c = x_k$ for some $k = 1, \cdots, n-1$. (excluding $x_0 = a$ & $x_n = b$)

Then $\dot{Q} = \{[x_{i-1}, x_i]; t_i\}_{i=1}^{k} \cup \{[x_{i-1}, x_i]; t_i\}_{i=k+1}^{n}$

Note that

$\dot{Q}_1 = \{[x_{i-1}, x_i]; t_i\}_{i=1}^{k}$ is a tagged partition of $[a, c]$ &

$\dot{Q}_2 = \{[x_{i-1}, x_i]; t_i\}_{i=k+1}^{n}$ is a tagged partition of $[c, b]$

Hence $S(f; \dot{Q}) = S(f_1; \dot{Q}_1) + S(f_2; \dot{Q}_2)$

Since $\|\dot{Q}_1\| \le \|\dot{Q}\| < \delta_\varepsilon \le \delta'$ &

$\|\dot{Q}_2\| \le \|\dot{Q}\| < \delta_\varepsilon \le \delta''$,

we have $|S(f_1; \dot{Q}_1) - L_1| < \varepsilon/3$

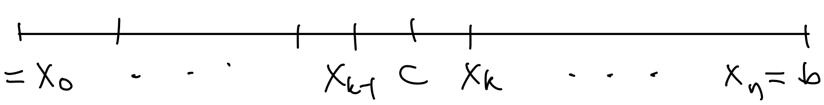
$|S(f_2; \dot{Q}_2) - L_2| < \varepsilon/3$.

Hence $|S(f; \dot{Q}) - (L_1 + L_2)|$

$\le |S(f_1; \dot{Q}_1) - L_1| + |S(f_2; \dot{Q}_2) - L_2|$

$< \frac{2\varepsilon}{3} < \varepsilon$.

Case (ii) $c \in (x_{k-1}, x_k)$ for some $k = 1, 2, \cdots, n$.



Then $[x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{k-2}, x_{k-1}] \cup [x_{k-1}, c]$

with tags $t_1 \in, t_2 \in, \cdots, t_{k-1} \in$, & $c \in$

is a tagged partition $\dot{Q}_1$ of $[a, c]$.

Similarly, $[c, x_k] \cup [x_k, x_{k+1}] \cup \cdots \cup [x_{n-1}, x_n]$ with tags
$\qquad\qquad\quad \overset{\cup\!\!\mathrm{l}}{c}\ , \qquad \overset{\cup\!\!\mathrm{l}}{t_{k+1}}\ , \ \cdots \qquad \overset{\cup\!\!\mathrm{l}}{t_n}$

is a tagged partition $\dot{Q}_2$ of $[c, b]$.

Then $S(f; \dot{Q})$

$$= \sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(t_k)(x_k - x_{k-1}) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= \left[\sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(c)(c - x_{k-1}) \right] - f(c)(c - x_{k-1})$$

$$+ f(t_k)(x_k - x_{k-1})$$

$$+ \left[f(c)(x_k - c) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1}) \right] - f(c)(x_k - c)$$

$$= S(f_1, \dot{Q}_1) - f(c)(c - x_{k-1}) + f(t_k)(x_k - x_{k-1})$$

$$+ S(f_2; \dot{Q}_2) - f(c)(x_k - c)$$

$$\Rightarrow \quad |S(f; \dot{Q}) - S(f_1, \dot{Q}_1) - S(f_2; \dot{Q}_2)|$$

$$\leq |f(t_k) - f(c)|\,|x_k - x_{k-1}|$$

$$\leq 2M \|\dot{Q}\| < 2M \cdot \frac{\varepsilon}{6M}$$

$$< \frac{\varepsilon}{3} \qquad \text{———} \ (*),$$

Also $\|\dot{Q}_1\| \leq \|\dot{Q}\|$ $\quad$ (as $0 < c - x_{k-1} < x_k - x_{k-1} \leq \|\dot{Q}\|$)

$\therefore \ \|\dot{Q}_1\| < \delta_\varepsilon < \delta'$

$$\Rightarrow \quad |S(f_1, \dot{Q}_1) - L_1| < \varepsilon/3 . \qquad \text{———} \qquad (*)_2$$

Similarly $\|\dot{Q}_2\| \leq \|\dot{Q}\| < \delta_\varepsilon < \delta''$

$$\Rightarrow \quad |S(f_2, \dot{Q}_2) - L_2| < \varepsilon/3 . \qquad \text{———} \qquad (*)_3$$

Then by $(*)_1$, $(*)_2$, & $(*)_3$

$$|S(f; \dot{Q}) - (L_1 + L_2)|$$

$$\leq |S(f; \dot{Q}) - S(f_1; \dot{Q}_1) - S(f_2; \dot{Q}_2)|$$

$$+ |S(f_1; \dot{Q}_1) - L_1| + |S(f_2; \dot{Q}_2) - L_2|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

This completes the proof of the Claim & hence the proof of the Thm. ✕✕

Cor 7.2.10 If $f \in \mathcal{R}[a,b]$ & $[c,d] \subset [a,b]$, then $f \in \mathcal{R}[c,d]$.

Pf: By Additivity Thm 7.2.9

$$f \in \mathcal{R}[a,b] \Rightarrow f \in \mathcal{R}[c,b] \Rightarrow f \in \mathcal{R}[c,d]$$ ✕✕

Cor 7.2.11 If $f \in \mathcal{R}[a,b]$ & $a = c_0 < c_1 < \cdots < c_m = b$,

then $f|_{[c_{i-1}, c_i]} \in \mathcal{R}[c_{i-1}, c_i]$ and

$$\int_a^b f = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} f$$

(Pf: By Induction)

Def: If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta \in [a,b]$ with $\alpha < \beta$,

we define $\int_\beta^\alpha f \overset{\text{def}}{=\!=} -\int_\alpha^\beta f$ and

$$\int_\alpha^\alpha f \overset{\text{def}}{=\!=} 0$$

Thm 7.2.13 If $f \in \mathcal{R}[a,b]$ and $\alpha, \beta, \gamma \in [a,b]$,

then $$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f \qquad \text{———} \quad (*)$$

in the sense that the existence of any two of these integrals exist implies the third integral exists & $(*)$ holds

Pf: If any two of α, β, γ equal, then $(*)$ is trivially holds (check)

If α, β, γ are distinct, we consider

$$L(\alpha,\beta,\gamma) \stackrel{\text{def}}{=} \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$$
$$= \int_\alpha^\beta f - \int_\gamma^\beta f - \int_\alpha^\gamma f$$

Clearly $L(\alpha,\beta,\gamma) = L(\beta,\gamma,\alpha) = L(\gamma,\alpha,\beta)$
$= -L(\alpha,\gamma,\beta) = -L(\gamma,\beta,\alpha) = -L(\beta,\alpha,\gamma)$

$$\Big(L(\alpha,\beta,\gamma) = \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f$$
$$= -\int_\beta^\alpha f - \int_\gamma^\beta f - \int_\alpha^\gamma f = -L(\alpha,\gamma,\beta) \Big)$$

By Additivity Thm 7.2.9,
if $\alpha<\gamma<\beta$, then $L(\alpha,\beta,\gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f\right) = 0$.

By the above, we have $L(\alpha,\beta,\gamma)=0$
for all other situations: $\gamma<\beta<\alpha$, $\beta<\alpha<\gamma$,
$\gamma<\alpha<\beta$, $\alpha<\beta<\gamma$, & $\beta<\gamma<\alpha$.

Hence $\forall\, \alpha,\beta,\gamma$,
$$0 = L(\alpha,\beta,\gamma) = \int_\alpha^\beta f - \left(\int_\alpha^\gamma f + \int_\gamma^\beta f \right)$$

i.e. $$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$$

$\#$