Case (b) $L = \pm\infty$. (of Thm 6.3.5)

subcase $L = +\infty$

$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = +\infty$ implies that

$\forall M > 1$, $\exists \delta > 0$ s.t. $\frac{f'(u)}{g'(u)} > M$

and hence $\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} > M$, $\forall\ a < \alpha < \beta < a + \delta$

As in case (a), $\lim_{x \to a+} g(x) = +\infty$ implies

$\exists$ c & $c_1 \in (a, a+\delta)$ such that

$$\begin{cases} \bullet\ a < c_1 < c < a + \delta \\ \bullet\ g(\alpha) > 0, \quad \forall \alpha \in (a, c] \\ \bullet\ 0 < \dfrac{g(c)}{g(\alpha)} < \dfrac{1}{2}, \quad \forall \alpha \in (a, c_1) \\ \bullet\ 0 \leq \dfrac{|f(c)|}{g(\alpha)} < \dfrac{1}{2}, \quad \forall \alpha \in (a, c_1) \end{cases}$$

Letting $\beta = c$, we have

$$\frac{f(c) - f(\alpha)}{g(c) - g(\alpha)} > M, \quad \forall \alpha \in (a, c)$$

And hence for $\alpha \in (a, c_1)$

$$\left(1 - \frac{g(c)}{g(\alpha)} > \frac{1}{2}\right)$$

$$\frac{f(\alpha) - f(c)}{g(\alpha)} > M\left(1 - \frac{g(c)}{g(\alpha)}\right) > \frac{1}{2}M, \quad \forall \alpha \in (a, c_1)$$

$$\Rightarrow \quad \frac{f(\alpha)}{g(\alpha)} > \frac{1}{2}M + \frac{f(c)}{g(\alpha)} > \frac{1}{2}M - \frac{1}{2}$$

$$= \frac{1}{2}(M-1), \qquad \forall \alpha \in (a, c_1)$$

Since $M>1$ is arbitrary, this shows that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = +\infty.$$

Subcase of "$L=-\infty$" is similar. $\#$

eg 6.3.6

(a) $\lim_{x\to\infty} \frac{\ln x}{x}$

- $f(x) = \ln x$ has derivative $f'(x) = \frac{1}{x}$ on $(0,\infty)$
- $g(x) = x$ has derivative $g'(x) = 1 \neq 0$ on $(0,\infty)$
- $\lim_{x\to\infty} g(x) = +\infty$
- $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

$\therefore$ L'Hospital Rule II $\Rightarrow$ $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.

(usually, one simply write $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$)

(b) $\lim_{x\to\infty} e^{-x}x^2 = \lim_{x\to\infty} \frac{x^2}{e^x}$

- $(x^2)' = 2x$, $\forall x$
- $(e^x)' = e^x \neq 0$ $\forall x$
- $e^x \to +\infty$ as $x \to +\infty$

But $\lim_{x\to\infty} \frac{2x}{e^x}$ still indeterminate.

So we need to start with $\lim_{x\to\infty} \frac{2x}{e^x}$ first:

- $(2x)' = 2$, $\forall x$
- $(e^x)' = e^x \neq 0$, $\forall x$
- $e^x \to +\infty$ as $x \to +\infty$
- $\lim_{x\to\infty} \frac{2}{e^x} = 0$ (exists)

$\therefore$ L'Hospital Rule $\Rightarrow$ $\lim_{x\to\infty} \frac{2x}{e^x} = 0$ (exists)

And applying L'Hospital Rule again, $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$.

(We usually just write

$$\lim_{x\to\infty} e^{-x}x^2 = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0 .)$$

(c) $\lim_{x\to 0+} \frac{\ln \sin x}{\ln x} = \lim_{x\to 0+} \frac{(\ln \sin x)'}{(\ln x)'}$

$= \lim_{x\to 0+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to 0^+} \cos x \cdot \frac{x}{\sin x}$

$= 1$ $\left(\text{as } \lim_{x\to 0+} \frac{x}{\sin x} = 1 = \lim_{x\to 0+} \cos x \right)$

(d) It is easy to see $\lim_{x\to\infty} \frac{x-\sin x}{x+\sin x} = \lim_{x\to\infty} \frac{1-\frac{\sin x}{x}}{1+\frac{\sin x}{x}} = 1.$

However, $\lim_{x\to\infty} \frac{(x-\sin x)'}{(x+\sin x)'} = \lim_{x\to\infty} \frac{1-\cos x}{1+\cos x}$ doesn't exist.

$\therefore$ The condition " $\lim_{x\to a+} \frac{f'(x)}{g'(x)}$ exists " is necessary for L'Hospital Rule.

Further examples (other indeterminate forms)

eg 6.3.7

(a) ($\infty - \infty$ form)

$$\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right) \qquad (x\in(0,\tfrac{\pi}{2}))$$

$$= \lim_{x\to 0+}\frac{\sin x - x}{x\sin x} \qquad \text{(transform to } \tfrac{0}{0} \text{ form)}$$

$$= \lim_{x\to 0+}\frac{\cos x - 1}{\sin x + x\cos x} \qquad \text{(L'Hospital) (still } \tfrac{0}{0} \text{ form)}$$

$$= \lim_{x\to 0+}\frac{-\sin x}{2\cos x - x\sin x} \qquad \text{(L'Hosptial)}$$

$$= 0 \qquad \text{(limit exists, calculation justified)}$$

(b) ($0\cdot(-\infty)$ form)

$$\lim_{x\to 0+} x\ln x \qquad (x\in(0,\infty))$$

$$= \lim_{x\to 0+}\frac{\ln x}{1/x} \qquad \text{(transforms to } \tfrac{-\infty}{\infty} \text{ form)}$$

$$= \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \qquad \text{(L'Hospital)}$$

$$= \lim_{x\to 0+}(-x) = 0 \qquad \text{(limit exists, calculation justified)}$$

(c) (0^0 form)

$$\lim_{x\to 0^+} x^x$$

$$= \lim_{x\to 0^+} e^{x\ln x}$$

$$= e^{\lim_{x\to 0^+} x\ln x}$$ (transforms to $0\cdot(-\infty)$ forms which can be calculated using L'Hospital as in (b))

$$= e^0 = 1$$

(d) (1^∞ form)

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \qquad x\in(1,\infty)$$

$$= \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$$

Transforms to the calculation of

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \qquad (\infty\cdot 0 \text{ form})$$

$$= \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} \qquad (\text{transform to } \tfrac{0}{0} \text{ form})$$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} \qquad (\text{L'Hospital})$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1 \qquad (\text{limit exists, calculation justified})$$

And hence $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})} = e$

(e) (∞^0 form)

$$\lim_{x\to 0+}\left(1+\frac{1}{x}\right)^x$$ $\left(x\in(0,\infty)\right)$ (limit of the other end)

$$= e^{\lim_{x\to 0+} x\ln\left(1+\frac{1}{x}\right)}$$ (transform to $0\cdot\infty$ form)

$$= e^{\lim_{x\to 0+}\frac{1}{1+\frac{1}{x}}}$$ (L'Hospital as before)

$$= e^0 = 1.$$ (limit exists, calculation justified)