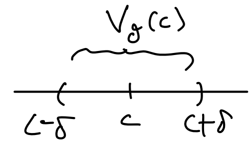
§6.2 The Mean Value Theorem

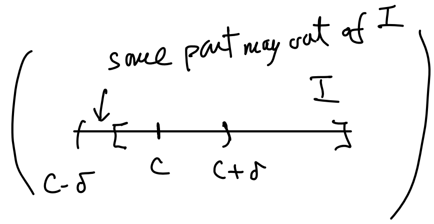
<u>Recall</u>: function $f: I \to \mathbb{R}$ is said to have a



- <u>relative maximum</u> at $c \in I$

  if $\exists$ a neighborhood of c, $V = V_\delta(c) = (c-\delta, c+\delta)$, such that

  $$f(x) \leq f(c), \quad \forall x \in V \cap I;$$



- <u>relative minimum</u> at $c \in I$

  if $\exists$ a neighborhood of c, $V = V_\delta(c) = (c-\delta, c+\delta)$, such that

  $$f(x) \geq f(c), \quad \forall x \in V \cap I;$$

- <u>relative extremum</u> at $c \in I$ if either "relative maximum" or "relative minimum".

<u>Thm 6.2.1</u> (<u>Interior Extremum Theorem</u>) (Same notations as above)

Let
- c be an <u>interior</u> point of the interval I
- f has a <u>relative extremum</u> at c.

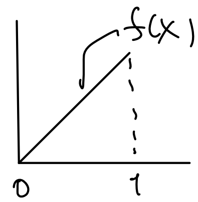
If $f'(c)$ exits, then $f'(c) = 0$.

<u>Note</u>: The condition that $c \in I$ is an interior point is <u>neccessary</u>:

<u>eg</u>: $f(x) = x$ on $[0,1]$ has relative extremum

at $x=0$ (min), but $f'(0) = 1 \neq 0$.

(at $x=1$ (max), but $f'(1) = 1 \neq 0$.)



Pf: Prove only the case of relative maximum. The case of relative minimum is similar.

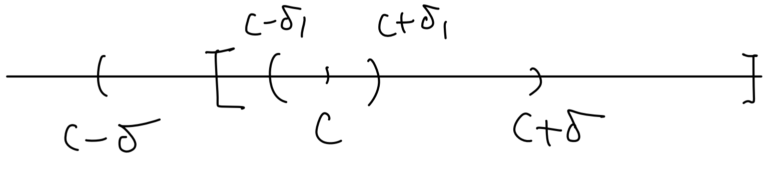
Let $c \in$ interior of I, f has a relative maximum at c and $f'(c)$ exists.

Suppose on the contrary that $f'(c) \neq 0$, then either $f'(c) > 0$ or $f'(c) < 0$.

If $f'(c) > 0$, i.e. $\lim\limits_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$.

Then (by Thm 4.2.9 of the Textbook, MATH2050), $\exists$ a nbd. $V = V_\delta(c)$ such that $\frac{f(x) - f(c)}{x - c} > 0 \quad \forall\ x \in V \cap I,\ x \neq c$.

Since $c \in$ interior of I, one can find a δ_1, $0 < \delta_1 < \delta$ (if needed) so that $(c - \delta_1, c + \delta_1) \subset V \cap I$.



Note that f has a relative maximum, there exists $\delta_2 > 0$ such that $f(x) \le f(c)$, $\forall\ x \in (c - \delta_2, c + \delta_2) \cap I$

Then for $\delta_3 = \min\{\delta_1, \delta_2\}$,

$(c - \delta_3, c + \delta_3) \subset V \cap I$ and

$(c - \delta_3, c + \delta_3) \subset (c - \delta_2, c + \delta_2) \cap I$

As a result,

$$\left.\begin{array}{l} \dfrac{f(x)-f(c)}{x-c} < 0, \\[2ex] \text{and } f(x) \le f(c) \end{array}\right\} \quad \forall\, x \in (c-\delta_3, c+\delta_3),\ x \neq c.$$

Since $(c, c+\delta_3) \subset (c-\delta_3, c+\delta_3) \subset V \cap I$

The 1^{st} inequality implies

$\exists\ x > 0$ in $(c-\delta_3, c+\delta_3)$ s.t.

$$\frac{f(x)-f(c)}{x-c} > 0 \Rightarrow f(x)-f(c) > 0,$$

which contradicts the 2^{nd} inequality.

Similarly, if $f'(c) < 0$, one can find $\delta_3' > 0$ so that

$$\left.\begin{array}{l} \dfrac{f(x)-f(c)}{x-c} < 0, \\[2ex] \text{and } f(x) \le f(c) \end{array}\right\} \quad \forall\, x \in (c-\delta_3', c+\delta_3'),\ x \neq c.$$

The 1^{st} inequality

$\Rightarrow \exists\ x < c$ such that $\dfrac{f(x)-f(c)}{x-c} < 0$.

$\Rightarrow \quad f(x)-f(c) > 0$ contradicts the 2^{nd} inequality.

$\therefore f'(c) = 0$. #

Cor 6.2.2 Let
- $f: I \to \mathbb{R}$ continuous
- f has a relative extremum at an interior point $c \in I$.

Then either
- $f'(c)$ doesn't exist

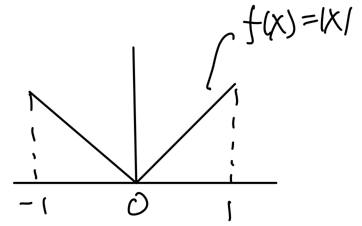
or
- $f'(c) = 0$.

(Pf: Follow easily from Thm 6.2.1)

eg: $f(x) = |x|$ on $I = [-1, 1]$.

interior minimum at $x = 0$.

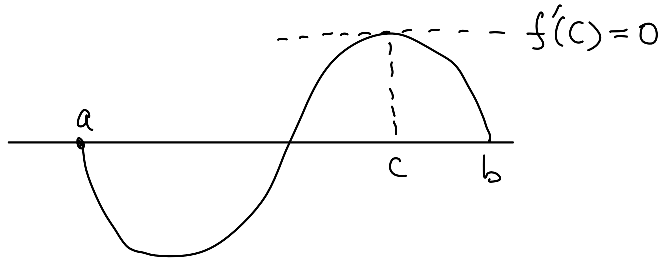
$f'(0)$ doesn't exist



Thm 6.2.3 (Rolle's Theorem)

Suppose
- $f: [a,b] \to \mathbb{R}$ continuous (on closed interval $I = [a,b]$) $(a<b)$
- $f'(x)$ exists $\forall\, x \in (a,b)$ (open interval, interior of I)
- $f(a) = f(b) = 0$

Then $\exists\ c \in (a,b)$ such that $f'(c) = 0$



Pf: If $f(x) \equiv 0$ on $[a,b]$, then $f'(x)=0 \ \forall \ x \in [a,b]$. We're done.

If $f(x) \not\equiv 0$, then either $f>0$ for some point in (a,b)

or $f<0$ for some point in (a,b).

Note that f is continuous on the closed interval $[a,b]$,

f attains an absolute maximum and an absolute minimum on I.

(Thm 5.3.4 of the Textbook, MATH2050)

Hence, if $f>0$ for some point in (a,b), f attains the absolute maximum, ie. the value $\sup\{f(x) : x \in I\} > 0$, at some point $c \in (a,b)$ as $f(a)=f(b)=0$.

Since $c \in (a,b)$, $f'(c)$ exists.

By Interior Extreme Theorem (Thm 6.2.1), $f'(c)=0$.

If there is no $x \in (a,b)$ s.t. $f>0$, then we must have

$f<0$ for some $x \in (a,b)$. Hence $(-f)>0$ for some $x \in (a,b)$

and $-f$ satisfies all conditions as f. Therefore,

$\exists \ c \in (a,b)$ such that $(-f)'(c)=0 \Rightarrow f'(c)=0$.

$\#$

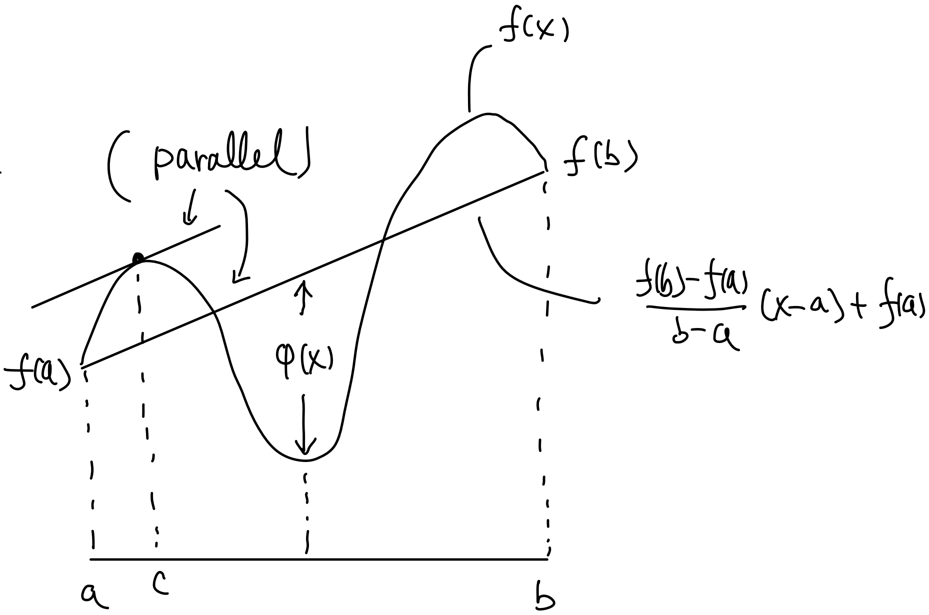
Thm 6.2.4 (Mean Value Theorem)

Suppose
- $f:[a,b]\to\mathbb{R}$ continuous $(a<b)$
- $f'(x)$ exists $\forall x\in(a,b)$

Then $\exists$ a point $c\in(a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a)$$

Pf: Consider the function defined on $[a,b]$:



$$\varphi(x)=f(x)-\left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right]$$

$$=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

Then φ is continuous on $[a,b]$ as f is continuous on $[a,b]$, and $\varphi'(x)$ exists $\forall x\in(a,b)$ as $f'(x)$ exists $\forall x\in(a,b)$.

At the end points

$$\varphi(a)=f(a)-f(a)-\frac{f(b)-f(a)}{b-a}(a-a)=0$$

$$\varphi(b)=f(b)-f(a)-\frac{f(b)-f(a)}{b-a}(b-a)=0$$

$\therefore$ φ satisfies all conditions in Rolle's Thm (Thm 6.2.3).

Hence $\exists$ $c \in (a,b)$ such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

(by Thm 6.1.3 and $(x)' = 1$)

$$\therefore \quad f(b) - f(a) = f'(c)(b-a). \quad ※$$

Applications of Mean Value Theorem

> **Thm 6.2.5** Suppose
> - $f : [a,b] \to \mathbb{R}$ continuous $(a<b)$
> - $f'(x)$ exists $\forall x \in (a,b)$ (ie. f differentiable on (a,b))
> - $f'(x) = 0$, $\forall x \in (a,b)$.
>
> Then f is a constant on $[a,b]$.

Pf: let $x \in [a,b]$ and $x > a$.

Applying Mean Value Thm to $f : [a,x] \to \mathbb{R}$,

(which clearly satisfies all conditions of the Thm)

we find a point $c \in (a,x)$ such that

$$f(x) - f(a) = f'(c)(x-a) = 0 \quad \text{(by assumption } f'(c)=0)$$

$$\Rightarrow \quad f(x) \equiv f(a), \ \forall x \in I.$$

$\therefore$ f is constant on I. ※

Cor 6.2.6 Suppose
- $f, g : [a,b] \to \mathbb{R}$ continuous
- f, g differentiable on (a,b)
- $f'(x) = g'(x),\ \forall x \in (a,b)$.

Then $\exists$ constant C such that $f = g + C$ on $[a,b]$.

Recall $f : I \to \mathbb{R}$ is said to be

- increasing on I if $x_1 < x_2\ (x_1, x_2 \in I) \Rightarrow f(x_1) \leq f(x_2)$
- decreasing on I if $-f$ is increasing on I.

Thm 6.2.7 Let $f : I \to \mathbb{R}$ be differentiable. Then

(a) f is increasing on $I \iff f'(x) \geq 0,\ \forall x \in I$

(b) f is decreasing on $I \iff f'(x) \leq 0,\ \forall x \in I$

Pf: (a) ($\Leftarrow$) Let $f'(x) \geq 0,\ \forall x \in I$.

Then for any $x_1, x_2 \in I$ with $x_1 < x_2$, we can apply the Mean Value Thm to $f : [x_1, x_2] \to \mathbb{R}$

(since f is differentiable on $I \Rightarrow f : [x_1, x_2] \to \mathbb{R}$ satisfies all conditions)

and find a point $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

$$\geq 0 \quad \text{since } f'(c) \geq 0 \ \& \ x_2 > x_1.$$

$\therefore f$ is increasing on I.

(a) ($\Rightarrow$) Suppose f is differentiable and increasing on I.

Then $\forall\, c \in I$, we have

$$\frac{f(x)-f(c)}{x-c} \geq 0 \ , \quad \forall\, x \in I,\ x \neq c$$

by "f is increasing" $\left(\begin{array}{l}\text{both "positive (or zero)" if } x>c, \\ \text{both "negative (or zero)" if } x<c\end{array}\right)$

Hence f differentiable at $c \Rightarrow$

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \geq 0$$

(b) Applying (a) to $-f$. ※

Remarks: (1) Strictly increasing : $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Then ex.13 of §6.2 $\Rightarrow$ "$f'(x)>0$ on $I \Rightarrow f$ is strictly increasing on I".

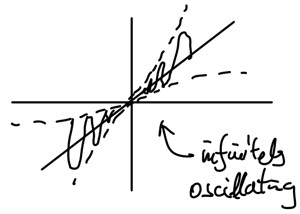
But :

"$f'(x)>0$ on $I \not\Leftarrow f$ is strictly increasing on I".

Counterexample : $f(x) = x^3 : \mathbb{R} \to \mathbb{R}$ is strictly increasing,

but $f'(0)=0$ which is not ">0".

(2) Consider $g(x) = \begin{cases} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.



Exercise 10 of §6.2 : $g'(0) = 1 > 0$, but $g(x)$ is not increasing in any neighborhood of 0.

(That is, $g'(x) > 0$ only at a point x is not sufficient to ensure $g(x)$ is increasing near the point. We need a whole interval!)

Thm 6.2.8 (First Derivative Test for Extrema)

Let
- $f : [a,b] \to \mathbb{R}$ continuous $(a < b)$
- $c \in (a,b)$
- f is differentiable on (a,c) and (c,b).

Then (a) If $\exists\, \delta > 0$ s.t. $\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c, c+\delta) \end{cases}$

then f has a relative maximum at c.

(b) If $\exists\, \delta > 0$ s.t. $\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c, c+\delta) \end{cases}$

then f has a relative minimum at c.

Pf: (a) If $x \in (c-\delta, c)$, then Mean Value Thm (applying to $f: [x,c] \to \mathbb{R}$) implies $\exists\, c_x \in (x,c)$ s.t.

$$f(c) - f(x) = f'(c_x)(c-x)$$
$$\geq 0 \quad (\text{since } f' \geq 0 \text{ on } (c-\delta, c))$$

If $x \in (c, c+\delta)$, then Mean Value Thm (applying to $f: [c,x] \to \mathbb{R}$) implies $\exists\, c_x \in (c,x)$ s.t.

$$f(x) - f(c) = f'(c_x)(x-c)$$
$$\leq 0 \quad (\text{since } f' \leq 0 \text{ on } (c, c+\delta))$$

Together we have $f(c) \geq f(x) \quad \forall\, x \in (c-\delta, c+\delta)$

$\therefore$ f has a relative maximum at c.

(b) Applying (a) to $-f$. ※

Remark: Converse of Thm 6.2.8 is not true.

i.e. $\exists$ differentiable function f has a relative maximum at c, but the statement

"$\exists\, \delta > 0$ s.t. $\begin{cases} \bullet\ (c-\delta, c+\delta) \subseteq [a,b] \\ \bullet\ f'(x) \geq 0 \text{ for } x \in (c-\delta, c) \\ \bullet\ f'(x) \leq 0 \text{ for } x \in (c, c+\delta) \end{cases}$ "

is not true (Exercise 9 of §6.2)

Further Applications of the Mean Value Theorem

Examples 6.2.9

(a) Rolle's Thm 6.2.3 can be used to "locate" roots of a function.

In fact, Rolle's Thm $\Rightarrow$

$g = f'$ always has a root between any two zeros of f

(provided f is differentiable & etc.)

explicit eg: $\left\{ \begin{array}{l} g(x) = \cos x = (\sin x)' \\ \sin x = 0 \text{ for } x = n\pi \text{ for } n \in \mathbb{Z}. \end{array} \right.$

Rolle's $\Rightarrow$ $\cos x$ has a root in $(n\pi, (n+1)\pi)$, $\forall n \in \mathbb{Z}$.

(eg. of Bessel functions J_n is omitted)

(b) Using Mean Value Theorem for approximate calculations & error estimates.

eg: Approximate $\sqrt{105}$.

Applying Mean Value Thm to $f(x) = \sqrt{x}$ on $[\overset{a}{\overset{\shortparallel}{100}}, \overset{b}{\overset{\shortparallel}{105}}]$,

$$f(105) - f(100) = f'(c)(105 - 100)$$

for some $c \in (100, 105)$.

In eg 6.1.10 (d), we've seen that $f'(c) = \frac{1}{2\sqrt{c}}$.

$\therefore$ $\sqrt{105} - \sqrt{100} = \frac{5}{2\sqrt{c}}$ for some $c \in (100, 105)$

$$\Rightarrow \quad 10+\frac{5}{2\sqrt{105}} < \sqrt{105} < 10+\frac{5}{2\sqrt{100}} = 10+\frac{5}{2\cdot 10} = 10.25$$

And $\sqrt{105} < \sqrt{121} = 11 \Rightarrow \sqrt{105} > 10+\frac{5}{2\cdot 11}$

Hence
$$\frac{205}{22} < \sqrt{105} < \frac{41}{4}$$

(Of course, the estimate can be improved by more care analysis)

Examples 6.2.10 (Inequalities)

(a) $e^x \geq 1+x$, $\forall x \in \mathbb{R}$ and "equality $\Leftrightarrow x=0$".

Pf: We will use the fact that

$f(x)=e^x$ has derivative $f'(x)=e^x$, $\forall x \in \mathbb{R}$

and $e^x > 1$ for $x>0$ (and $f'(0)=1$)

$e^x < 1$ for $x<0$.

(To be defined and proved in §8.3.)

If $x=0$, then $e^x = 1 = 1+x$. We're done.

If $x>0$, applying MVT (Mean Value Thm) to

$f(x)=e^x$ on $[0,x]$,

we have $c \in (0,x)$ such that

$$e^x - e^0 = e^c(x-0)$$

$$\therefore \quad e^x - 1 > x .$$

If $x<0$, applying MVT to $f(x)=e^x$ on $[x,0]$,

we have $c\in(x,0)$ such that

$$e^0-e^x=e^c(0-x)$$

$$1-e^x<-x \qquad (e^c<1,\ -x>0)$$

$$\therefore \quad e^x>1+x \ ,\ \forall\ x<0.$$

Finally, one observes, in both cases, the inequality is strict. So "equality $\Leftrightarrow x=0$". ※

(b) $-x\le \sin x\le x \ ,\ \forall\, x\ge 0.$

Pf: The inequalities are clear for $x=0$.

Let $x>0$. Consider $g(x)=\sin x$ on $[0,x]$.

Then MVT implies $\exists\ c\in(0,x)$ s.t.

$$\sin x-\sin 0=(\cos c)(x-0)$$

Using $-1\le\cos c\le 1$ and $\sin 0=0$, we have

$$-x\le\sin x\le x \qquad (\text{as } x>0)$$ ※