Inverse function

Thm 6.1.8 Let
- $I \subseteq \mathbb{R}$ be an interval
- $f: I \to \mathbb{R}$ be strictly monotone and continuous.
- $J = f(I)$ and $g: J \to \mathbb{R}$ be the strictly monotone & continuous function inverse to f.

If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d = f(c)$ and

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$$

(Note $f'(c) \neq 0$ doesn't follow from f being strictly monotone:
eg. $f(x) = x^3$ is strictly monotone, but $f'(0) = 0$.
In this case, the inverse $g(x) = x^{\frac{1}{3}}$ is not differentiable at $x=0$.)

Pf: Since f is differentiable at $x = c$, Carathéodory's Thm 6.1.5
$\Rightarrow \exists\ \varphi: I \to \mathbb{R}$ with φ continuous at c such that

$$\begin{cases} f(x) - f(c) = \varphi(x)(x-c), \ \forall x \in I, \text{ and} \\ \varphi(c) = f'(c) \end{cases}$$

Since $f'(c) \neq 0$ and φ is continuous at c, $\exists\ \delta > 0$ such that

$$\varphi(x) \neq 0, \ \forall x \in (c-\delta, c+\delta) \cap I.$$

Let $U = f((c-\delta, c+\delta)\cap I) \subset J$

Then the inverse function g satisfies $f(g(y)) = y$, $\forall y \in U$.

Hence $y - d = f(g(y)) - f(c) = \varphi(g(y))(g(y)-c)$

$$= \varphi(g(y))(g(y)-g(d)) \qquad \left(\begin{array}{l} d = f(c) \in U \\ \Rightarrow c = g(d) \end{array}\right)$$

Since $g(y) \in (c-\delta, c+\delta)\cap I$, $\forall y \in U$,

we have $\varphi(g(y)) \neq 0$.

Hence $$g(y) - g(d) = \frac{1}{\varphi(g(y))}(y-d).$$

Since g is continuous on J and φ is continuous at $c = g(d)$ & $\neq 0$,

$1/\varphi\circ g$ is continuous at d.

Then by Carathéodory's Thm 6.1.5, g is differentiable at $d = f(c)$

and $$g'(d) = \frac{1}{\varphi(g(d))} = \frac{1}{\varphi(c)} = \frac{1}{f'(c)}. \quad \#$$

<u>Thm 6.1.9</u> (Same notations as in Thm 6.1.8)

Let $f: I \to \mathbb{R}$ be strict monotone (no need to assume continuity).

If f is differentiable on I and $f'(x) \neq 0$, $\forall x \in I$. Then the inverse function g is differentiable on $J = f(I)$ and

$$g' = \frac{1}{f'\circ g}$$

Pf: f diff. on $I \Rightarrow f$ is continuous. Then apply Thm 6.1.8 to all $x \in I$. #

Remark on simplified notations:

Usually, we write $y = f(x)$ and $x = g(y)$ for inverse functions to each other. Then the formula in Thm 6.1.9 can be written as

$$g'(y) = \frac{1}{(f' \circ g)(y)} \quad \forall y \in J$$

or

$$(g' \circ f)(x) = \frac{1}{f'(x)}, \quad \forall x \in I$$

In this notation, one often simply write

$$g'(y) = \frac{1}{f'(x)}$$

without explicitly stated that $y = f(x)$ & $x = g(y)$!

eg 6.1.10

(a) $f(x) = x^5 + 4x + 3$ gives a strictly increasing (why?) and continuous function on $\mathbb{R}$ (and $f(\mathbb{R}) = \mathbb{R}$ why?)

$f'(x) = 5x^4 + 4 \geq 4 > 0$.

Therefore, Thm 6.1.8 $\Rightarrow g = f^{-1}$ is differentiable $\forall y \in \mathbb{R}$.

And for example, at $x = 1$, $g'(8) = g'(f(1)) = \frac{1}{f'(1)} = \frac{1}{9}$

(b) $f : [0,\infty) \to [0,\infty)$ given by $f(x) = x^n$ where $n = 2, 4, 6, \cdots$

Then f is strictly increasing continuous on $[0,\infty)$

Note that $f([0,\infty)) = [0,\infty)$. The inverse function g defines on $[0,\infty)$ and is strictly increasing and continuous.

Since $f'(x) = nx^{n-1} > 0$, $\forall x > 0$, & $f((0,\infty)) = (0,\infty)$, g is differentiable $\forall y > 0$ and

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{n(g(y))^{n-1}} = \frac{1}{n(y^{\frac{1}{n}})^{n-1}} = \frac{1}{n} y^{\frac{1}{n}-1}$$

(The inverse is denoted by $g(y) = y^{\frac{1}{n}}$, $\forall y \in [0,\infty)$.)

Note: g is not differentiable at $y=0$ (one side derivative doesn't exist. Omitted!. But the argument is the same as in the next example.)

(C) $n = 3, 5, 7, \cdots$. $F(x) = x^n$, $\forall x \in \mathbb{R}$, is strictly increasing & continuous.

Inverse is $G(y) = y^{\frac{1}{n}}$, $\forall y \in \mathbb{R}$.

As in example (b) above, G is differentiable $\forall y \neq 0$

and $\quad G'(y) = \frac{1}{n} y^{\frac{1}{n}-1}$ $\quad$ (check!)

And again, G is not differentiable at $y=0$

Pf Suppose that G is differentiable at $y=0$.

Then consider the composite function $y = F(G(y))$.

Since $G(0)=0$ and $F'(0)=0$ exists.

Chain rule implies $1 = \frac{dy}{dy} = \underbrace{F'(G(0))}_{=0} \underbrace{G'(0)}_{\text{exists}} = 0$

which is a contradiction. $\therefore$ $G'(0)$ doesn't exist ※

(d) Recall if $r = \frac{m}{n} > 0$, $m, n \in \{1,2,3,\cdots\}$, then

$$x^r = x^{\frac{m}{n}} \text{ is defined as } (x^{\frac{1}{n}})^m, \quad \forall x \geq 0.$$

Therefore, the function $R: [0,\infty) \to [0,\infty)$ defined by

$$R(x) = x^r, \quad \forall x \geq 0$$

is a composite function $R = f \circ g$ where

$$g(x) = x^{\frac{1}{n}}, \quad x \geq 0 \qquad \text{(the inverse discussed in eg (b))}$$

and $$f(x) = x^m, \quad x \geq 0$$

(i.e. $R(x) = x^r = (x^{\frac{1}{n}})^m = f(g(x)), \ \forall x \in [0,\infty)$)

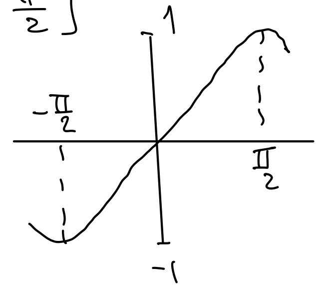
Then Chain rule $\Rightarrow$ $\forall x \in [0,\infty)$

$$R'(x) = f'(g(x))\, g'(x) = m\,(x^{\frac{1}{n}})^{m-1} \frac{1}{n} x^{\frac{1}{n}-1}$$

$$= \left(\frac{m}{n}\right) x^{\left(\frac{m}{n}\right)-1}$$

$\therefore$ $(x^r)' = r x^{r-1}$, $\forall x \geq 0$, true for all rational $r > 0$.

(e) $\sin x$ is strictly increasing on $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and maps I to $J = [-1, 1]$.



$\Rightarrow$ inverse exists, and we denote it by

$$\text{Arcsin} : [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

i.e. If $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ & $y \in [-1, 1]$, then

$$y = \sin x \iff x = \text{Arcsin}\, y.$$

Note that $D \sin x = \cos x \neq 0$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (no end pts.)

Thm 6.1.8 $\Rightarrow$

$$D\,\text{Arcsin}\, y = \frac{1}{D \sin x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}}$$

$$= \frac{1}{\sqrt{1 - y^2}}, \quad \forall y \in (-1, 1)$$

(Note: $D\,\text{Arcsin}\, y$ does not exist for $y = \pm 1$. Check!)