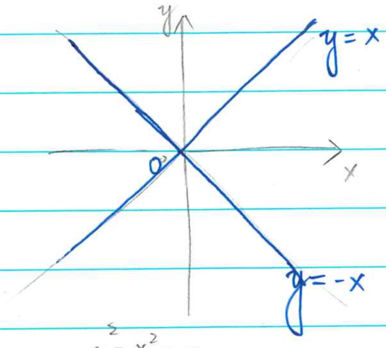
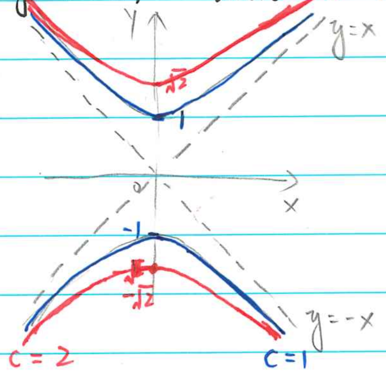
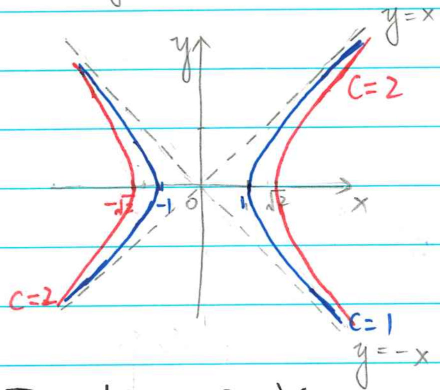
1. (a) Sketch the curves $y^2 - x^2 = c$, where $c = 0, 1, 2, -1, -2$



$y^2 - x^2 = 0$





For asymptotes: (Consider $x \to \infty$ for Explanation)

$$k_1 = \lim_{x\to\infty} \frac{\sqrt{x^2+c}}{x} = 1$$

$$k_2 = \lim_{x\to\infty} \frac{-\sqrt{x^2+c}}{x} = -1$$

$$\lim_{x\to\infty} \left(\sqrt{x^2+c} - (k_1 x + b_1)\right) = 0$$

$$\Rightarrow \quad b_1 = 0$$

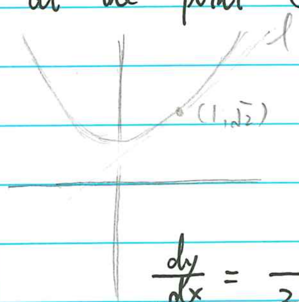
$$\lim_{x\to\infty} \left(-\sqrt{x^2+c} - (k_2 x + b_2)\right) = 0 \quad \Rightarrow \quad b_2 = 0$$

Remarks: ① You can consider $\begin{cases} y \geq 0 \\ y < 0 \end{cases}$ and $\begin{cases} x \geq 0 \\ x < 0 \end{cases}$ for sketching.

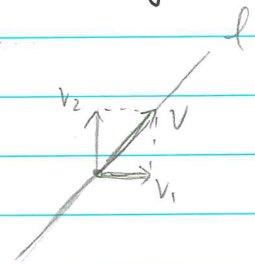
② Be aware of the asymptotes!

(As $x \to \infty$ or $x \to -\infty$, the curve behaves like a straight line!)

(b) Find the vector tangent to the curve $y = \sqrt{c + x^2}$, $c = 1$, at the point $(1, \sqrt{2})$



Note:



$\frac{v_2}{v_1}$ = slope of l

$$\frac{dy}{dx} = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$k = \text{slope of } l = \left.\frac{dy}{dx}\right|_{x=1} = \frac{\sqrt{2}}{2}$$

Let $\vec{v} = (v_1, v_2)$ be a tangent vector.

$$\frac{v_2}{v_1} = k \quad \Rightarrow \quad v_1 = \sqrt{2}\, v_2$$

Take $v_2 = 1$, then $\vec{v} = (\sqrt{2}, 1)$

(usually we can do normalization : $|\vec{v}| = \sqrt{v_1^2 + v_2^2} = 1$

$\Rightarrow v_1 = \frac{2\sqrt{3}}{3}, \quad v_2 = \frac{\sqrt{3}}{3}$)

(c) $f(x, y) = y^2 - x^2$, show that $\nabla f \perp v$ (at $(1, \sqrt{2})$)

$f_x = -2x, \quad f_y = 2y$

$\Rightarrow \nabla f(x, y) = (-2x \quad 2y) = f_x \hat{i} + f_y \hat{j} \quad \left(= (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}) f \right)$

~~$\nabla f(1, \sqrt{2}) =$~~ $= -2x\hat{i} + 2y\hat{j}$

$\nabla f(1, \sqrt{2}) = -2\hat{i} + 2\sqrt{2}\hat{j} = (-2 \quad 2\sqrt{2})$

$\nabla f(1, \sqrt{2}) \cdot \vec{v} = (-2, \; 2\sqrt{2}) \cdot (\sqrt{2} \; 1) = -2\sqrt{2} + 2\sqrt{2} = 0$

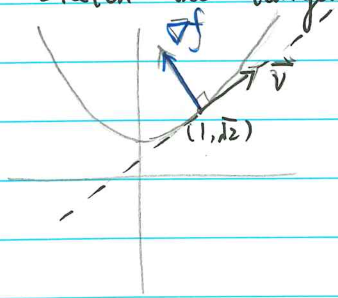
or : $\nabla f(1, \sqrt{2}) \cdot \vec{v} = (-2\hat{i} + 2\sqrt{2}\hat{j}) \cdot (\sqrt{2}\hat{i} + \hat{j})$

$= -2\sqrt{2} \; \hat{i} \cdot \hat{i} + 4 \hat{j} \cdot \hat{i} + (-2) \hat{i} \cdot \hat{j} + 2\sqrt{2} \; \hat{j} \cdot \hat{j}$

$= -2\sqrt{2} + 0 + 0 + 2\sqrt{2} = 0$.

$\Rightarrow \nabla f(1, \sqrt{2}) \perp \vec{v}$

(d) Sketch the tangent vector and ∇f



2. $\nabla (\frac{f}{g}) = ?$

For ~~simplifi~~ simplicity, assume $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (depends on 2 variables)

$\nabla (\frac{f}{g}) = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j})(\frac{f}{g}) = \frac{\partial}{\partial x}(\frac{f}{g})\hat{i} + \frac{\partial}{\partial y}(\frac{f}{g})\hat{j}$

Note : $\frac{\partial}{\partial x}(\frac{f}{g}) = \frac{f_x}{g} - \frac{f \cdot g_x}{g \cdot g}$

$\frac{\partial}{\partial y}(\frac{f}{g}) = \frac{f_y}{g} - \frac{f \cdot g_y}{g \cdot g}$

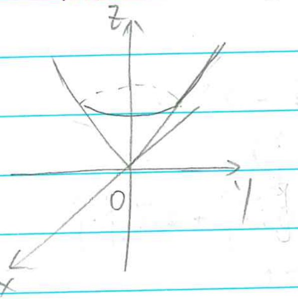
So $\nabla\left(\frac{f}{g}\right) = \left(\frac{f_x}{g} - \frac{f \cdot g_x}{g^2}\right)\hat{i} + \left(\frac{f_y}{g} - \frac{f \cdot g_y}{g^2}\right)\hat{j}$

$= \frac{f_x\hat{i} + f_y\hat{j}}{g} - \frac{f}{g^2}\cdot(g_x\hat{i} + g_y\hat{j})$

$= \frac{\nabla f}{g} - \frac{f \cdot \nabla g}{g^2} = \frac{g \cdot \nabla f - f\nabla g}{g^2}$

(The idea is the same for higher dimensional case $f, g : \mathbb{R}^n \to \mathbb{R}$)

3. Sketch $z = \sqrt{x^2+y^2}$



4. Sketch $\cancel{z = \sqrt{x^2+y^2}}$ $z = xy$

(In each direction e.g. fix x. $z = ky$, a st. line)
Think of a line gradually goes down and up)
You are advised to search for the picture.