MATH5011 Real Analysis I

Exercise 8 Suggested Solution

For those who have learnt functional analysis, this exercise serves to refresh your memory. For those who have not learnt it, working through the problems gives you some feeling on the subject.

(1) Provide two proofs that $C[0,1]$ is an infinite dimensional vector space.

Solution:

First proof. It is clear that $\{x^n : n = 0, 1, \dots\}$ forms a basis for the subspace $P[0,1] \subset C[0,1]$ of polynomials on $[0,1]$. Hence $\dim C[0,1] \geq \dim P[0,1] = \infty$.

Second proof. With reference to the lecture notes, $C[0,1]$ does not have the Heine-Borel Property. Theorem 4.1 then implies $\dim C[0,1] = \infty$.

(2) Show that both $C_c(0,1)$ and $C^1(0,1)$ are not closed subspaces in $C[0,1]$ and hence they are not Banach space.

Solution: For $C_c(0,1)$, we consider $f_n, n > 1$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (0, \frac{1}{2n}), \\ \text{linear} & \text{if } x \in [\frac{1}{2n}, \frac{1}{n}), \\ \frac{1}{2} - |x - \frac{1}{2}| & \text{if } x \in [\frac{1}{n}, 1 - \frac{1}{n}), \\ \text{linear} & \text{if } x \in [1 - \frac{1}{n}, 1 - \frac{1}{2n}), \\ 0 & \text{if } x \in [1 - \frac{1}{2n}, 1). \end{cases}$$

Obviously, $f_n \to \frac{1}{2} - |x - \frac{1}{2}|$ uniformly on $(0,1)$ and hence $C_c(0,1)$ is not closed.

For $C^1(0,1)$, we consider the following example:

$$f_n(x) = \sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}}.$$

with

$$f_n'(x) = \frac{(x-\frac{1}{2})}{\sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}}}.$$

and both are continuous on $[0,1]$. Obviously $f_n(x) \to f(x) := |(x-1/2)|$ pointwisely with the limit does not belong to $C^1[0,1]$. Moreover

$$\begin{aligned} |f - f_n(x)| &= |\sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}} - \sqrt{(x-\frac{1}{2})^2}| \\ &= |\frac{\frac{1}{n}}{\sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}} + \sqrt{(x-\frac{1}{2})^2}}| \\ &\le \frac{\frac{1}{n}}{\sqrt{(x-\frac{1}{2})^2 + \frac{1}{n}}} \\ &\le \frac{1}{\sqrt{n}} \end{aligned}$$

Hence $\|f - f_n\|_\infty \to 0$. and the $C^1(0,1)$ is not closed

(3) Endow $C[0,1]$ with the norm $\|f\| = \int_0^1 |f(x)|dx$. Determine whether it is complete or not.

Solution: The space is not complete, we consider $f_n, n > 1$,

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2} - \frac{1}{n}), \\ \text{linear} & \text{if } x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}], \\ -1 & \text{if } x \in [\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

$$\forall m > n, \|f_m - f_n\| < \frac{2}{n} \to 0.$$

Then$\{f_n\}$ is Cauchy and obviously there is no $f \in C[0,1]$ s.t.$\|f - f_n\| \to 0$.

(4) Let Λ be a bounded linear functional on the normed space X. Show that its operator norm

$$\begin{aligned}\|\Lambda\|_{op} &= \sup\left\{\frac{\Lambda x}{\|x\|} : x \neq 0\right\} \\ &= \inf\{M : |\Lambda x| \leq M\|x\|,\ \forall x \in X\}.\end{aligned}$$

Solution: To prove the first equality, note that

$$\|\Lambda\|_{op} = \sup\left\{\max\left(\frac{\Lambda x}{\|x\|}, \frac{\Lambda(-x)}{\|-x\|}\right) : x \neq 0\right\} = \sup\left\{\frac{\Lambda x}{\|x\|} : x \neq 0\right\}.$$

For the second, we have $|\Lambda x| \leq \|\Lambda\|_{op}\|x\|$, which implies

$$\|\Lambda\|_{op} \geq \inf\{M : |\Lambda x| \leq M\|x\|,\ \forall x \in X\}.$$

Also, if M has $|\Lambda x| \leq M\|x\|$, $\forall x \in X$, then $\dfrac{|\Lambda x|}{\|x\|} \leq M$, which gives $\|\Lambda\|_{op} \leq M$. Taking inf on both sides, we have

$$\|\Lambda\|_{op} \leq \inf\{M : |\Lambda x| \leq M\|x\|,\ \forall x \in X\}.$$

(5) For any normed space $(X, \|.\|)$, prove that $(X', \|.\|_{op})$ forms a Banach space.

Solution: It is clear that X' is a vector space and $\|\cdot\|_{op}$ is a norm on X'. It suffices to prove the completeness.

Suppose $\{\Lambda_n\}$ is Cauchy in $(X', \|\cdot\|_{op})$, i.e.

$$\forall \varepsilon > 0,\ \exists N \text{ such that } \forall m, n \geq N,\ \|\Lambda_m - \Lambda_n\|_{op} < \varepsilon.$$

For any $x \in X$, the inequality $\|\Lambda_m x - \Lambda_n x\| \leq \|\Lambda_m - \Lambda_n\|_{op}\|x\|$ shows that

$\{\Lambda_n x\}$ is Cauchy in the scalar field, hence convergent. Define Λ by $\Lambda x = \lim_{n\to\infty} \Lambda_n x$. It is straightforward to verify that Λ is bounded, linear and, in view of

$$\|\Lambda_n - \Lambda\|_{op} = \sup_{\|x\|=1} |\Lambda_n x - \Lambda x| \to 0 \quad \text{as } n \to \infty,$$

we conclude that the dual space of a normed space is always complete. The completeness is in fact inherited from the completeness of the scalar field $\mathbb{R}$.

(6) Let X be a Hilbert space and X_1 a proper closed subspace. For x_0 lying outside X_1, let $d = \|x_0 - z\|$ where d is the distance from x_0 to X_1. Show that

$$\langle x, z - x_0 \rangle = 0, \quad \forall x \in X_1.$$

Hint: For $x \in X_1$, one has $\frac{d}{dt}\phi(t) = 0$ at $t = 0$ where $\phi(t) = \|z_0 + tx - x_0\|^2$. Why?

Solution: Since $\phi(t)$ attains its minimum at $t = 0$, we have $\phi'(0) = 0$. It is easy to see that

$$\begin{aligned}\phi'(t) &= \frac{d}{dt}\langle z_0 + tx - x_0, z_0 + tx - x_0 \rangle \\ &= 2\langle x, z_0 + tx - x_0 \rangle.\end{aligned}$$

Putting $t = 0$ yields the result.

(7) Show that the correspondence $\Lambda \mapsto w$ in Theorem 4.8 is norm preserving.
Solution: By Cauchy-Schwarz inequality, $\forall x \in X$,

$$|\Lambda(x)| = |\langle x, w \rangle| \le \|x\|\|w\|$$

With equality holds when $x = w$. Hence $\|\Lambda\|_{op} = \|w\|$ and the map is norm preserving.

(8) Let Λ_1 and Λ_2 be two bounded linear functionals on the Hilbert space X. Suppose that they have the same kernel. Prove that there exists a nonzero constant c such that $\Lambda_2 = c\Lambda_1$. Use this fact to give a proof of Theorem 4.8

Solution: We may suppose kernel of Λ_1 and Λ_2 is a proper subspace of X and $\exists x_0 \in X, \Lambda_1(x_0), \Lambda_2(x_0) \neq 0$, then $\forall x \in X$,

$$\begin{aligned}\Lambda_1\big(x - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}x_0\big) &= \Lambda_1(x) - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}\Lambda_1(x_0) \\ &= 0.\end{aligned}$$

As the two functionals have the same kernel, we have

$$\begin{aligned}\Lambda_2\big(x - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}x_0\big) &= \Lambda_2(x) - \frac{\Lambda_1(x)}{\Lambda_1(x_0)}\Lambda_2(x_0) \\ &= 0.\end{aligned}$$

Hence

$$\Lambda_2 = \frac{\Lambda_2(x_0)}{\Lambda_1(x_0)}\Lambda_1.$$

Now Let Λ be a non zero bounded linear functional on X and x_0 not in $ker\Lambda$, then $\exists z \in ker\Lambda$ s.t.

$$\langle x, x_0 - z\rangle = 0, \forall x \in ker\Lambda.$$

Theorem 4.8 follows by letting $\Lambda(x) = \Lambda_2(x)$ and $\langle x, x_0 - z\rangle = \Lambda_1(x)$.

(9) This is optional. Read Page 23 and on in [SS] for the following striking application of the Hahn-Banach theorem:

There exists $m : \mathcal{P}_{\mathbb{R}} \to [0, \infty]$ satisfying

(1) $m(E_1 \cup E_2) = m(E_1) + m(E_2),\ E_1, E_2 \subset \mathbb{R},\ E_1 \cap E_2 = \emptyset$,

(2) $m(E) = \mathcal{L}^1(E)$ whenever E is $\mathcal{L}^1$-measurable,

(3) $m(E + a) = m(E)$, $\forall E \subset \mathbb{R}$, $\forall a \in \mathbb{R}$.

Of course, m cannot be countably additive.

Solution: See Theorem 5.6, Stein and R. Shakarchi, Functional Analysis.