MATH5011

Exercise 4 Suggested Solution

(1) Identify the Riesz measures corresponding to the following positive functionals ($X = \mathbb{R}$):

(a) $\Lambda_1 f = \displaystyle\int_a^b f\,dx$, and

(b) $\Lambda_2 f = f(0)$.

Solution:

(a) $\mu_1 =$ the restriction of the Lebesgue measure on $[a,b]$.

$\mu_1(E) = \mathcal{L}^1(E \cap [a,b])$

(b) The Dirac delta measure at 0.

(2) Let c be the counting measure on $\mathbb{R}$,

$$c(A) = \begin{cases} \#A, & A \neq \phi, \\ 0, & A = \phi. \end{cases}$$

Is there a positive functional

$$\Lambda f = \int f\,dc \quad ?$$

Solution: No, let f(x) be a non-negative continuous function of compact support that is 1 for all x in $[0,1]$ and decreases to zero outside the interval,

$$\int f dc \geq \int \chi_{[0,1]} dc = \infty.$$

(3) Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \text{ if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \text{ if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, \dots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy.$$

Let μ be the measure associated with this Λ by Theorem 2.14 in [R]. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

Solution: Write $p_i = (x_i, y_i)$, $i = 1, 2$. Denote

$$d(p_1, p_2) = \begin{cases} |y_1 - y_2|, & x_1 = x_2, \\ 1 + |y_1 - y_2|, & x_1 \neq x_2. \end{cases}$$

We prove that d is a metric.

- $d(p_1, p_2) \geq 0$ and $d(p_1, p_2) = 0$ if and only if $p_1 = p_2$.
- $d(p_1, p_2) = d(p_2, p_1)$.
- $d(p_1, p_2) \leq d(p_1, p_3) + d(p_3, p_2)$ holds because $|y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2|$.

Now we claim that (X, τ) is a locally compact Hausdorff space. Let τ_1 be the discrete topology on $\mathbb{R}$, so every singleton $\{x\}$ is an open set. Then every point $x \in \mathbb{R}$ has the compact set $\{x\}$ as a neighborhood, so that $(\mathbb{R}, \tau_1)$ is a locally compact Hausdorff space. Note that $(X, \tau) = (\mathbb{R}, \tau_1) \times (\mathbb{R}, \tau_2)$, where τ_2 is the usual topology of $\mathbb{R}$. The claim follows.

If K is compact in X, the first projection $\mathrm{pr}_1(K)$ is compact in $(\mathbb{R}, \tau_1)$. Hence it is a finite set. Therefore K is a finite union

$$\{x_1\} \times K_1 \cup \cdots \cup \{x_n\} \times K_n,$$

where each K_i, $i = 1, 2, \ldots, n$, is a compact set in $(\mathbb{R}, \tau_2)$.

If $f : X \to \mathbb{C}$ has compact support, then $\mathrm{spt} f \subset \{x_1, \ldots, x_n\} \times \mathbb{R}$. Thus,

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy$$

defines a positive linear functional on $C_c(X)$.

By the proof of Riesz's representation theorem, the measure μ defined by the equalities

$$\begin{aligned} \mu(V) &= \sup_{K \subset V \text{ compact}} \mu(K) = \sup_{f \prec V} \Lambda f, \\ \mu(E) &= \inf_{V \supset E \text{ open}} \mu(V) \end{aligned}$$

is a representing measure for Λ. Using the second equality with the Lebesgue measure m on $\mathbb{R}$, we observe that

$$\mu(\{x\} \times K) = m(K).$$

Thus μ is characterized by the identity

$$\mu(\{x\} \times [a, b]) = b - a, \qquad x \in \mathbb{R}.$$

Let V be an open set containing $\mathbb{R} \times \{0\}$. Then for $x \in \mathbb{R}$, $(x, 0) \in V$, so that there exists an $\varepsilon_x > 0$ with

$$\{x\} \times [-\varepsilon_x, \varepsilon_x] \subset V.$$

This implies that there must be an n with uncountably many $\varepsilon_x \geq 1/n$. (Otherwise, $\varepsilon_x \geq 1/n$ for at most countably many x, contradicting the fact that $\mathbb{R}$ is uncountable.)

Let

$$K_x = \{x\} \times \left[-\frac{\varepsilon_x}{2}, \frac{\varepsilon_x}{2}\right], \quad \varepsilon_x \geq \frac{1}{n}.$$

For $K = \bigcup_{j=1}^{m} K_{x_j}$, we have $\mu(K) \geq \frac{m}{n}$. Hence, if $V \supset \mathbb{R} \times \{0\}$ is open, then $\mu(V) \geq \sup_{m \in \mathbb{N}} \frac{m}{n} = \infty$. This implies $\mu(\mathbb{R} \times \{0\}) = \infty$.

Now if K is a compact subset of $\mathbb{R} \times \{0\}$, then $K = \{x_1, \ldots, x_n\} \times \{0\}$, which implies $\mu(K) = 0$.

Therefore for $E = \mathbb{R} \times \{0\}$, $\mu(E) = \infty$ while $\sup_{K \subset E \text{ compact}} \mu(K) = 0$. This means that μ is not inner regular.

(4) Let λ be a Borel measure and μ a Riesz measure on $\mathbb{R}^n$ such that $\lambda(G) = \mu(G)$ for all open sets G. Show that λ coincides with μ on $\mathcal{B}$.

Solution: Let $E \in \mathcal{B}$. For $\varepsilon > 0$, by Proposition 2.10, there exists an open set E and a closed set F with $F \subset E \subset G$ such that $\mu(G \setminus F) < \varepsilon$. Since G and $G \setminus F$ are open, λ and μ coincide on them, and one has

$$\begin{aligned}\mu(E) &= \mu(G) - \mu(G \setminus E) \geq \mu(G) - \mu(G \setminus F) = \lambda(G) - \lambda(G \setminus F) \\ &\geq \lambda(E) - \varepsilon.\end{aligned}$$

By changing the position of μ and λ, one has

$$\lambda(E) - \varepsilon \leq \mu(E) \leq \lambda(E) + \varepsilon.$$

Since this holds for any $\varepsilon > 0$, one has $\mu(E) = \lambda(E)$.

(5) Let μ be a Borel measure on $\mathbb{R}^n$ such that $\mu(K) < \infty$ for all compact K. Show that μ is the restriction of some Riesz measure on $\mathcal{B}$. Hint: Use Riesz

representation theorem and Problem 4. This exercise gives a characterization of the Riesz measure on $\mathbb{R}^n$.

Solution: Let λ be the Riesz measure associated with the functional

$$\Lambda f = \int f d\mu, \quad \forall f \in C_c(X),$$

from the Riesz representation theorem. Let G be an open set. Then

$$\lambda(G) = \sup\{\Lambda f : f < G\}.$$

By Urysohns lemma, it follows that $\lambda(G) = \mu(G)$. Then apply Problem 4.

(6) Let μ be a Riesz measure on $\mathbb{R}^n$. Show that for every measurable function f, there exists a sequence of continuous function $\{f_n\}$ such that $f_n \to f$ almost everywhere.

Solution: As μ is finite on compact set, $\{x \in \mathbb{R}^n : |f(x)| < \infty, |x| \leq m\}$ is a μ finite measurable set, by theorem 2.12, let $f_m = f \cdot \chi_{\{x \in R^n : |f(x)| < \infty, |x| \leq m\}}$, $\exists g_m$ continuous on $\mathbb{R}^n$ s.t

$$\mu(\{x : f_m(x) \neq g_m(x)\}) \leq \frac{1}{2^m}.$$

As $\sum_{m=1}^{\infty} \mu(\{x : f_m(x) \neq g_m(x)\}) < \infty$, by Borel-Cantelli Lemma, $\exists \mu$ null set N s.t.

$$\forall x \in R^n \setminus N, \exists K, \forall m \geq K, f_m(x) = g_m(x).$$

$\{g_m\}$ obviously converges to $f\chi_{\{x \in R^n : |f(x)| < \infty\}}$m a.e.. Similarly, we obtain sequences of continuous function $\{h_m\}$ and $\{j_m\}$ by replacing f_m by $m \cdot$

$\chi_{\{x\in R^n: f(x)=\infty, |x|\le m\}}$ and $-m\cdot\chi_{\{x\in R^n: f(x)=-\infty, |x|\le m\}}$, we have

$$g_m + h_m + j_m \to f a.e..$$

(7) A *step function* on $\mathbb{R}$ is a simple function s where $s^{-1}(a)$ is either empty or an interval for every $a \in \mathbb{R}$. Show that for every Lebesgue integrable function f on $\mathbb{R}$, there exists a sequence of step functions $\{s_j\}$ such that

$$\lim_{j\to\infty}\int \big|s_j(x) - f(x)\big| d\mathcal{L}^1(x) = 0.$$

Hint: Approximate f by simple functions (see Ex 2) and then apply Lusin's theorem.

Solution: Without loss of generality, we may assume f is non-negative, it suffices to show for all $\varepsilon > 0$, there exists step function ψ s.t.

$$\int \big|\psi(x) - f(x)\big| d\mathcal{L}^1(x) < \varepsilon.$$

There exists non-negative simple function $s(x)$ of compact support s.t. $s(x) \le f(x)$ and

$$\int \big|s(x) - f(x)\big| d\mathcal{L}^1(x) < \frac{\varepsilon}{3}.$$

Let $M = \max_{x\in\mathbb{R}} |s(x)|$, apply Lusin's theorem to $s(x)$, we have a continuous g of compact support, for instance $\subseteq [a,b]$ s.t.

$$\mathcal{L}^1(\{s \neq g\}) < \frac{\varepsilon}{6M+1}$$

and

$$\sup_{x\in\mathbb{R}} |g(x)| \le M.$$

Therefore

$$\int \big|g(x) - s(x)\big| d\mathcal{L}^1(x) = \int_{\{s \neq g\}} \big|g(x) - s(x)\big| d\mathcal{L}^1(x) < \frac{\varepsilon}{3}.$$

The function g is obviously Riemann integrable over $[a, b]$ and the Riemann integral of g is the same as its Lebesgue integral, so $\exists$ step function $\psi \leq g$ s.t

$$\int \big|g(x) - \psi(x)\big| d\mathcal{L}^1(x) < \frac{\varepsilon}{3},$$

and

$$\begin{aligned} \int \big|f(x) - \psi(x)\big| d\mathcal{L}^1(x) &\leq \int f(x) - s(x) + \big|g(x) - s(x)\big| + g(x) - \psi(x) d\mathcal{L}^1(x) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$