MATH5011 Suggested Solution to Exercise 2

(1) Let f be a non-negative measurable function.

(a) Prove Markov's inequality:

$$\mu\Big\{x \in X : \ f(x) \geq M\Big\} \leq \frac{1}{M} \int_X f d\mu,$$

for all $M > 0$.

(b) Deduce that every integrable function is finite a.e..

(c) Deduce that $f = 0$ a.e. if f is integrable and $\int f = 0$.

Solution:

(a) let $F = \{x : f(x) > M\}$, then by non-negativity of f, we have

$$M\mu(F) \leq \int_F f \, d\mu \leq \int_X f \, d\mu,$$

and Markov's inequality follows.

(b) Let $E_n = \{x : f(x) > n\}$ and $E = \{x : f(x) = \infty\}$. Clearly we have E_n is descending and $\bigcap_{n=1}^{\infty} E_n = E$. So, $\forall n \in N$, by Markov's inequality, we have $\mu(E_1) \leq \int_X f\,d\mu < \infty$ and

$$\begin{aligned}
\mu(E) &= \lim_{n\to\infty} \mu(E_n) \\
&= \lim_{n\to\infty} \int_{E_n} d\mu \\
&\leq \lim_{n\to\infty} \frac{1}{n} \int_{E_n} f\,d\mu \\
&\leq \lim_{n\to\infty} \frac{1}{n} \int_X f\,d\mu \\
&= \int_X f\,d\mu \lim_{n\to\infty} \frac{1}{n} \\
&= 0,
\end{aligned}$$

since $\int_X f\,d\mu$ is finite. Thus, f is finite a.e..

(c) Let $E_n = \left\{x : f(x) \geq \frac{1}{n}\right\}$ and $E = \{x : f(x) > 0\}$. Then E_n ascends to E. We have

$$\mu(E_n) = \int_{E_n} d\mu \leq n \int_{E_n} f\,d\mu = 0,\ \forall n \in N.$$

Therefore,

$$\mu(E) = \lim_{n\to\infty} \mu(E_n) = 0.$$

(2) Let g be a measurable function in $[0, \infty]$. Show that

$$m(E) = \int_E g\,d\mu$$

defines a measure on $\mathcal{M}$. Moreover,

$$\int_X f\,dm = \int_X fg\,d\mu, \qquad \forall f \text{ measurable in } [0, \infty].$$

Solution: We readily check that

(1) $m(\emptyset) = 0$;

(2) $m(E) \geq 0$, $\forall E \in M$;

(3) For mutually disjoint $A_k \in M$,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \int_X \sum_{k=1}^{\infty} \chi_{A_k} g \, d\mu = \sum_{k=1}^{\infty} \int \chi_{A_k} g \, d\mu = \sum_{k=1}^{\infty} m(A_k)$$

by monotone convergence theorem, since $\sum_{k=1}^{n} \chi_{A_k} g \uparrow \sum_{k=1}^{\infty} \chi_{A_k} g$.

To prove the last assertion, consider the following cases:

(a) $f = \chi_E$ for some $E \in M$.

$$\int_X f \, dm = \int_E dm = m(E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu = \int_X fg \, d\mu.$$

(b) f is a non-negative simple function.

This follows from (a).

(c) f is a non-negative measurable function.

Pick a sequence $s_n \geq 0$ of simple functions such that $s_n \uparrow f$ pointwisely. Then $0 \leq s_n g \uparrow g$ pointwisely. From (b),

$$\int_X s_n \, dm = \int_X s_n g \, d\mu.$$

Taking $n \to \infty$, by monotone convergence theorem, we have

$$\int_X f \, dm = \int_X fg \, d\mu.$$

(3) Let $\{f_k\}$ be measurable in $[0, \infty]$ and $f_k \downarrow f$ a.e., f measurable and $\int f_1 \, d\mu < \infty$. Show that

$$\lim_{k \to \infty} \int f_k \, d\mu = \int f \, d\mu.$$

What happens if $\displaystyle\int f_1 \, d\mu = \infty$?

Solution: Without loss of generality, we may suppose $f_k \downarrow f$ pointwisely. (Otherwise, replace by X by $Y = X \setminus N$, such that $\mu(N) = 0$ and $f_k \downarrow f$ on Y.) Then $0 \le f_1 - f_k \uparrow f_1 - f$. By monotone convergence theorem,

$$\lim_{k\to\infty} \int_X (f_1 - f_k) \, d\mu = \int_X (f_1 - f) \, d\mu.$$

Since $\displaystyle\int_X f_1 \, d\mu < \infty$, we can cancel it from both sides to yield the result.

If $\int_X f_1 \, d\mu = \infty$, the result does not hold. For example, one may take $X = R$, $f_k(x) = 1/k$ and $f = 0$. Then

$$\int_X f \, d\mu = 0, \text{ while } \int_X f_k \, d\mu = \infty, \ \forall k \in N.$$

(4) Let f be a measurable function. Show that there exists a sequence of simple functions $\{s_j\}$, $|s_1| \le |s_2| \le |s_3| \le \cdots$, and $s_k(x) \to f(x)$, $\forall x \in X$.

Solution: Choose sequences of non-negative simple functions $s_j^+ \uparrow f_+$ and $s_j^- \uparrow f_-$. Put $s_j = s_j^+ \chi_{\{x: f(x) \ge 0\}} - s_j^- \chi_{\{x: f(x) < 0\}}$. Fix $x \in X$. If $f(x) \ge 0$ then $|s_j(x)| = s_j^+(x) \uparrow f_+$. If $f(x) < 0$ then $|s_j(x)| = s_j^-(x) \uparrow f_-$. We also have

$$s_j(x) \to f_+ \chi_{\{x: f(x) \ge 0\}}(x) - f_- \chi_{\{x: f(x) < 0\}}(x) = f(x), \quad \forall x \in X.$$

(5) Let $\mu(X) < \infty$ and f be integrable. Suppose that

$$\frac{1}{\mu(E)} \int_E f \, d\mu \in [a, b], \ \forall E \in \mathcal{M}, \mu(E) > 0$$

for some $[a, b]$. Show that $f(x) \in [a, b]$ a.e..

Solution: Let $A = \{x : f(x) < a\}$ and $B = \{x : f(x) > b\}$. If $\mu(A) > 0$, then

$$\frac{1}{\mu(A)}\int_A f\,d\mu < \frac{1}{\mu(A)}\int_A a\,d\mu = a,$$

a contradiction. Thus, $\mu(A) = 0$ and, similarly, $\mu(B) = 0$. Hence, $f(x) \in [a, b]$ a.e.

(6) Let f be Lebsegue integrable on $[a, b]$ which satisfies

$$\int_a^c f d\mathcal{L}^1 = 0,$$

for every c. Show that f is equal to 0 a.e..

Solution: Using Problem 2, we can define two measures m_+, m_- on $[a, b]$ by

$$m_+(E) := \int_E f_+ d\mathcal{L}^1, \quad m_-(E) := \int_E f_- d\mathcal{L}^1$$

Using $\int_a^c f d\mathcal{L}^1 = m_+((a, c)) - m_-((a, c)) = 0$, one sees that $m_+(I) = m_-(I)$, for every open interval $I \subset [a, b]$. Since every open set can be represented as a countable union of disjointed intervals, one has that $m_+(O) = m_-(O)$, for every open set $O \subset [a, b]$. Since Borel sets are generated by open sets, this holds for every Borel, and hence measurable sets E. This shows that

$$\int_E f d\mathcal{L}^1 = m_+(E) - m_-(E) = 0.$$

Setting $E = \{x \in [a, b] : f(x) \geq= 0\}$, one has $f_+ = 0$. Similarly $f_- = 0$. Hence $f = 0$ a.e.

(7) Let $f \geq 0$ be integrable and $\int f\,d\mu = c \in (0, \infty)$. Prove that

$$\lim_{n\to\infty} \int n \log\left(1 + \left(\frac{f}{n}\right)^{\alpha}\right) d\mu = \begin{cases} \infty, & \text{if } \alpha \in (0, 1) \\ c, & \text{if } \alpha = 1 \\ 0, & \text{if } 1 < \alpha < \infty. \end{cases}$$

Solution: Let $g_n(x) = n \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right)$. Since $\int f\,d\mu = c \in (0, \infty)$, we know that $\mu(\{x : f(x) = \infty\}) = 0$ and $\mu(\{x : f(x) > 0\}) > 0$. Observe that

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \infty, & \text{on } \{x : f(x) > 0\}, \text{ if } \alpha < 1, \\ f(x), & \text{a.e. } \mu, \text{ if } \alpha = 1, \\ 0, & \text{a.e. } \mu, \text{ if } \alpha > 1. \end{cases}$$

Moreover, if $\alpha \geq 1$, using the inequalities $1 + x^\alpha \leq (1+x)^\alpha$ and $\log(1+x) \leq x$ for $x \geq 0$, we have

$$g_n \leq n \log\left(1 + \frac{f}{n}\right)^\alpha \leq n\alpha \cdot \frac{f}{n} = \alpha f \in L^1(\mu).$$

- Suppose $\alpha \in (0, 1)$. By Fatou's lemma,

$$\varliminf_{n\to\infty} \int g_n\,d\mu \geq \int \varliminf_{n\to\infty} g_n\,d\mu = \infty.$$

  Hence, $\lim_{n\to\infty} \int g_n\,d\mu = \infty$.

- Suppose $\alpha = 1$. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int g_n\,d\mu = \int \lim_{n\to\infty} g_n\,d\mu = \int f\,d\mu = c.$$

- Suppose $1 < \alpha < \infty$. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int g_n\,d\mu = \int \lim_{n\to\infty} g_n\,d\mu = 0.$$

(8) Let f be a non-negative integrable function with respect to some μ and let $F_k = \{x : f(x) \geq k\}$ for $k \geq 1$. Show that $\sum_k \mu(F_k) < \infty$. Hint: Relate F_k to $E_k = \{x : k \leq f(x) \leq k+1\}$.

Solution: We can write $F_k = \bigcup_{n=k}^\infty E_n$, where $E_k = \{x : k \leq f(x) < k+1\}$

are pair-wise disjoint. Hence

$$\mu(F_k) = \sum_{n=k}^{\infty} \mu(E_n),$$

and one has, by changing the order of summations,

$$\begin{aligned} \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \mu(E_n) \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mu(E_n) = \sum_{n=1}^{\infty} n\mu(E_n) \\ \leq \sum_{n=1}^{\infty} \int_{E_n} f d\mu \leq f d\mu < \infty. \end{aligned}$$

(9) Let $\mu(X) < \infty$ and $f_k \to f$ uniformly on X and each f_k is bounded. Prove that

$$\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu.$$

Can $\mu(X) < \infty$ be removed?

Solution: We assume that $\mu(X) > 0$. (Otherwise, the result is trivial.) Let $\varepsilon > 0$ be given. Since $f_k \to f$ uniformly on X, there exists natural number N such that for all $k \geq N$ and for all $x \in X$, we have

$$|f_k(x) - f(x)| < \frac{\varepsilon}{\mu(X)}.$$

So, for all $k \geq N$, we have

$$\left| \int f_k \, d\mu - \int f \, d\mu \right| \leq \int |f_k - f| \, d\mu < \varepsilon.$$

The result follows.

If $\mu(X) = \infty$, the result no longer holds. One may take $X = R$, $f_k(x) = 1/k$, $f(x) = 0$ and μ to be the Lebesgue measure. Then $f_k \to f$ uniformly on X

and each f_k is bounded,

$$\int f\,d\mu = 0, \text{ while } \int f_k\,d\mu = \infty, \ \forall k.$$

(10) Give another proof of Borel-Cantelli lemma (in Ex.1) by using Corollary 1.12. (Hint: Study $g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x)$.)

Solution: Let $\{A_k\}$ be measurable, $A = \{x \in X : x \in A_k \text{ for infinitely many } k\}$ and suppose $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Write

$$g(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x).$$

Then $x \in A$ if and only if $g(x) = \infty$. By Fatou's lemma,

$$\int g\,d\mu \leq \sum_{j=1}^{\infty} \int \chi_{A_j}\,d\mu = \sum_{j=1}^{\infty} \mu(A_j) < \infty.$$

Now,

$$\mu(A) = \frac{1}{n}\int_A n\,d\mu \leq \frac{1}{n}\int_A g\,d\mu \leq \frac{1}{n}\int g\,d\mu.$$

Taking $n \to \infty$, we have $\mu(A) = 0$.

(11) Give an example of a sequence $\{f_k\}$ on $[0,1]$, $f_k \to f$ in L^1 with respect to $\mathcal{L}^1$ but it does not converge at any point in $[0,1]$.

Hint: Divide $[0,1]$ into $2^k, k \geq 1$, many subintervals of equal length and order them in a sequence. Let $I_j^k, j = 1, 2, \cdots, 2^k$, be these subintervals and consider the sequence composed of the characteristic functions of I_j^k.

Solution: Define f_n on $[0,1]$ as follows. Given $n \in N$, write $n = 2^k + m$ where $k = k(n) \geq 0$ and $m = m(n) \in \{0, 1, \ldots, 2^k - 1\}$. Then

$$f_n = \chi_{[m\cdot 2^{-k}, (m+1)\cdot 2^{-k}]}$$

is as required.

Clearly, $\displaystyle\int f_n \, d\mu = 2^{-k(n)} \to 0$ as $n \to \infty$, so $f_n \to 0$ in $L^1(\mu)$.

On the other hand, since $f_n(x) = a_n$, the sequence $\{f_n(x)\}$ does not converge at $x \in [0,1]$ except those with expansion $0.a_1a_2a_3\ldots$ where a_n's become 0 after some digit. But there are countably many such x's and we can redefine f_n at these points so that $\{f_n\}$ also diverges at them.

(12) Let f be a Riemann integrable function on $[a,b]$ and extend it to $\mathbb{R}$ by setting it zero outside $[a,b]$.

(a) Show that f is Lebsegue measurable.

(b) Show that the Riemann integral of f is equal to $\int_{\mathbb{R}} f d\mathcal{L}^1$.

(c) Give an example of a sequence of Riemann integrable functions which is uniformly bounded on $[a,b]$ and converges pointwisely to some function which is not Riemann integrable.

Solution:

(a) We assume the result and notation in question 10 of exercise 1, by the proof of 10b), f is Riemann integrable on $[a,b]$ if and only if $\overline{R}(f) = \underline{R}(f)$. When this holds, $L = \overline{R}(f) = \underline{R}(f)$. Then for all natural number n, we may find partition of [a,b], $P_n = \{a = z_0 < z_1 < ... < z_{m_n} = b\}$ such that

$$0 \leq \overline{R}(P_n, f) - \underline{R}(P_n, f) \leq \frac{1}{n},$$

define two sequence of step function in the following way, for all x in $[z_j, z_{j+1})$,

$$\varphi_n(x) = \inf \left\{ f(x) : x \in [z_j, z_{j+1}] \right\} ,$$

and

$$\psi_n(x) = \sup \left\{ f(x) : x \in [z_j, z_{j+1}] \right\}.$$

For all x in $[a, b]$

$$h(x) = \sup \left\{ \varphi_n(x) : n \in N \right\}$$

and

$$g(x) = \inf \left\{ \psi_n(x) : n \in N \right\} ,$$

h and g are obviously Lebesgue measurable, we also have $\varphi_n(x) \le h \le f \le g \le \psi_n(x)$. For any natural number n,

$$0 \le \int_a^b (g - h) d\mathcal{L}^1 \le \int_a^b (\psi_n - \varphi_n) d\mathcal{L}^1 = \overline{R}(P_n, f) - \underline{R}(P_n, f) \le \frac{1}{n} ,$$

so we have $h = f = g$ a.e. and f is Lebesgue measurable.

(b) By taking refinement with the partition $\{a = z_0 < z_1 = a + (b - a)/n < .. < z_j = a + j(b - a)/n < .. < z_{m_n} = b\}$ if necessary, we may assume the norm of partition P_n in (a) tend to 0 as $n \to 0$. As φ_n and ψ_n are integrable and $|f(x)| \le |\varphi_n(x)| + |\psi_n(x)|$ for all x in $[a, b]$, f is Lebesgue integrable and

$$\underline{R}(P_n, f) = \int_a^b \varphi_n d\mathcal{L}^1 \le \int_a^b f d\mathcal{L}^1 \le \int_a^b \psi_n d\mathcal{L}^1 = \overline{R}(P_n, f) .$$

Using result in 10(b) of Ex.1 and let n go to ∞, we have Riemann integral $= \int_R f d\mathcal{L}^1$.

(c) We consider the famous Dirichlet function g which is not Riemann integrable, $g(x)$ =1 if x is rational and $\in [0, 1]$, $g(x)$ =0 otherwise. Let $\{q_n : n \in N\}$ be an enumeration of all rational number in $[0, 1]$ and define

$$f_n = \sum_{i=1}^{n} \chi_{q_i} .$$

Then each f_n is obviously uniformly bounded Riemann integrable with zero integral and yet $\{f_n\}$ converges pointwisely to the Dirichlet function for all x in $[0, 1]$.