MATH5011 Exercise 9

(1) Optional. Let $\mathfrak{M}$ be the collection of all sets E in the unit interval $[0,1]$ such that either E or its complement is at most countable. Let μ be the counting measure on this σ-algebra $\mathfrak{M}$. If $g(x) = x$ for $0 \leq x \leq 1$, show that g is not $\mathfrak{M}$-measurable, although the mapping

$$f \mapsto \sum x f(x) = \int fg \, d\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \neq L^\infty$ in this situation.

(2) Optional. Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0,1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$, and therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fg \, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

(3) Prove Brezis-Lieb lemma for $0 < p \leq 1$.

Hint: Use $|a+b|^p \leq |a|^p + |b|^p$ in this range.

(4) Let $f_n, f \in L^p(\mu)$, $0 < p < \infty$, $f_n \to f$ a.e., $\|f_n\|_p \to \|f\|_p$. Show that $\|f_n - f\|_p \to 0$.

(5) Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \ldots$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p \, d\mu < C$ for all n. Prove that

$$\lim_{n\to\infty} \int_X |f - f_n| \, d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

(6) We have the following version of Vitali's convergence theorem. Let $\{f_n\} \subset L^p(\mu)$, $1 \le p < \infty$. Then $f_n \to f$ in L^p-norm if and only if

(i) $\{f_n\}$ converges to f in measure,

(ii) $\{|f_n|^p\}$ is uniformly integrable, and

(iii) $\forall \varepsilon > 0$, $\exists$ measurable E, $\mu(E) < \infty$, such that $\displaystyle\int_{X\setminus E} |f_n|^p \, d\mu < \varepsilon$, $\forall n$.

I found this statement from PlanetMath. Prove or disprove it.

(7) Let $\{x_n\}$ be bounded in some normed space X. Suppose for Y dense in X', $\Lambda x_n \to \Lambda x$, $\forall \Lambda \in Y$ for some x. Deduce that $x_n \rightharpoonup x$.

(8) Consider $f_n(x) = n^{1/p}\chi(nx)$ in $L^p(\mathbb{R})$. Then $f_n \rightharpoonup 0$ for $p > 1$ but not for $p = 1$. Here $\chi = \chi_{[0,1]}$.

(9) Let $\{f_n\}$ be bounded in $L^p(\mu)$, $1 < p < \infty$. Prove that if $f_n \to f$ a.e., then $f_n \rightharpoonup f$. Is this result still true when $p = 1$?

(10) Provide a proof of Proposition 5.3.

(11) Show that $M(X)$, the space of all signed measures defined on $(X, \mathfrak{M})$, forms a Banach space under the norm $\|\mu\| = |\mu|(X)$.

(12) Let $\mathcal{L}^1$ be the Lebesgue measure on $(0, 1)$ and μ the counting measure on $(0, 1)$. Show that $\mathcal{L}^1 \ll \mu$ but there is no $h \in L^1(\mu)$ such that $d\mathcal{L}^1 = h \, d\mu$. Why?

(13) Let μ be a measure and λ a signed measure on $(X, \mathfrak{M})$. Show that $\lambda \ll \mu$ if and only if $\forall \varepsilon > 0$, there is some $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ whenever $|\mu(E)| < \delta$, $\forall E \in \mathfrak{M}$.

(14) Let μ be a σ-finite measure and λ a signed measure on $(X, \mathfrak{M})$ satisfying $\lambda \ll \mu$. Show that

$$\int f \, d\lambda = \int fh \, d\mu, \quad \forall f \in L^1(\lambda), \ fh \in L^1(\mu)$$

where $h = \dfrac{d\lambda}{d\mu} \in L^1(\mu)$.

(15) Let μ, λ and ν be finite measures, $\mu \gg \lambda \gg \nu$. Show that $\dfrac{d\nu}{d\mu} = \dfrac{d\nu}{d\lambda}\dfrac{d\lambda}{d\mu}$, μ a.e.