MATH5011 Real Analysis I

Exercise 8

Standard notations are in force. Those with *,taken from [R], are optional.

(1)

$$\Phi(t) = \int_X |f + tg|^p \, d\mu$$

is differentiable at $t = 0$ and

$$\Phi'(0) = p \int_X |f|^{p-2} f g \, d\mu.$$

Hint: Use the convexity of $t \mapsto |f + tg|^p$ to get

$$|f + tg|^p - |f|^p \le t(|f + g|^p - |f|^p), \quad t > 0$$

and a similar estimate for $t < 0$.

(2) Suppose f is a measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(b) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$

(3) Assume, in addition to the hypothesis of the previous problem, that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left\{ \int_X \log|f| \, d\mu \right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

(4) For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ is $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

(5) * Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f \, d\mu \cdot \int_\Omega g \, d\mu \geq 1.$$

(6) * Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h \, d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} \, d\mu \leq 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

(7) * Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x} \int_0^x f(t) \, dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0$ a.e.

(c) Prove that the constant $\dfrac{p}{p-1}$ cannot be replaced by a smaller one.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (a) Assume first that $f \geq 0$ and $f \in C_c((0, \infty))$. Integration by parts gives

$$\int_0^\infty F^p(x) \, dx = -p \int_0^\infty F^{p-1}(x) x F'(x) \, dx.$$

Note that $xF' = f - F$, and apply Hölder's inequality to $\displaystyle\int F^{p-1} f$. Then derive the general case.

(c) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. See also Exercise 14, Chap. 8 in [R].

(8) * Consider $L^p(\mathbb{R}^n)$ with the Lebesgue measure, $0 < p < \infty$. Show that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ holds $\forall f, g$ implies that $p \geq 1$. Hint: For $0 < p < 1$, $x^p + y^p \geq (x + y)^p$.

(9) * Consider $L^p(\mu)$, $0 < p < 1$. Then $\dfrac{1}{q} + \dfrac{1}{p} = 1$, $q < 0$.

(a) Prove that $\|fg\|_1 \geq \|f\|_p \|g\|_q$.

(b) $f_1, f_2 \geq 0$. $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.

(c) $d(f, g) \overset{\text{def}}{=\!=} \|f - g\|_p^p$ defines a metric on $L^p(\mu)$.

(10) Give a proof of the separability of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, without using Weierstrass approximation theorem.

Suggestion: Cover $\mathbb{R}^n$ with many cubes and consider the combinations $s = \sum \alpha_j \chi_{C_j}$ where C_j are the cubes and $\alpha_j \in \mathbb{Q}$.

(11) (a) Let X_1 be a subset of the metric space (X, d). Show that (X_1, d) is separable if (X, d) is separable.

(b) Let $E \subset \mathbb{R}^n$ and consider $L^p(E)$, $1 \leq p < \infty$, where the measure is understood to be the restriction of $\mathcal{L}^n$ on E. Is it separable?

(12) Let X be a metric space consisting of infinitely many elements and μ a Borel measure on X such that $\mu(B) > 0$ on any metric ball (i.e. $B = \{x : d(x, x_0) < \rho\}$ for some $x_0 \in X$ and $\rho > 0$. Show that $L^\infty(\mu)$ is non-separable.

Suggestion: Find disjoint balls $B_{r_j}(x_j)$ and consider $\chi_{B_{r_j}(x_j)}$.

(13) Show that $L^1(\mu)' = L^\infty(\mu)$ provided $(X, \mathfrak{M}, \mu)$ is σ-finite, i.e., $\exists X_j$, $\mu(X_j) < \infty$, such that $X = \bigcup X_j$.

Hint: First assume $\mu(X) < \infty$. Show that $\exists g \in L^q(\mu)$, $\forall q > 1$, such that

$$\Lambda f = \int fg \, d\mu, \quad \forall f \in L^p, \; p > 1.$$

Next show that $g \in L^\infty(\mu)$ by proving the set $\{x : |g(x)| \geq M + \varepsilon\}$ has measure zero $\forall \varepsilon > 0$. Here $M = \|\Lambda\|$.

(14) (a) For $1 \leq p < \infty$, $\|f\|_p, \|g\|_p \leq R$, prove that

$$\int \big||f|^p - |g|^p\big| \, d\mu \leq 2pR^{p-1} \|f - g\|_p .$$

(b) Deduce that the map $f \mapsto |f|^p$ from $L^p(\mu)$ to $L^1(\mu)$ is continuous.

Hint: Try $|x^p - y^p| \leq p|x - y|(x^{p-1} + y^{p-1})$.