MATH5011 Exercise 6

Many problems are taken from [R].

(1) Show that $\mathcal{C} = \{x \in [0,1] : x = 0.a_1a_2a_3\cdots,\ a_j \in \{0,2\}$ in one of its ternary expansion.$\}$.

(2) Let $0 < \varepsilon < 1$. Construct an open set $G \subset [0,1]$ which is dense in $[0,1]$ but $\mathcal{L}^1(E) = \varepsilon$.

(3) Here we construct a Cantor-like set, or a Cantor set with positive measure, with positive measure by modifying the construction of the Cantor set as follows. Let $\{a_k\}$ be a sequence of positive numbers satisfying

$$\gamma \equiv \sum_{k=1}^{\infty} 2^{k-1} a_k < 1.$$

Construct the set $\mathcal{S}$ so that at the kth stage of the construction one removes 2^{k-1} centrally situated open intervals each of length a_k. Establish the facts:

(a) $\mathcal{L}^1(\mathcal{S}) = 1 - \gamma$,

(b) $\mathcal{S}$ is perfect,

(c) $\mathcal{S}$ is uncountable.

(4) Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $\mathcal{L}^1(A)$.

(5) Let $\mathcal{N}$ be a Vitali set in $[0,1]$. Show that $\mathcal{M} = [0,1] \setminus \mathcal{N}$ has measure 1 and hence deduce that

$$\mathcal{L}^1(\mathcal{N}) + \mathcal{L}^1(\mathcal{M}) > \mathcal{L}^1(\mathcal{N} \cup \mathcal{M}).$$

Remark: A Vitali set is the non-measurable set constructed in Notes. I have no idea what $\mathcal{L}^1(\mathcal{N})$ is, except that it is positive.

(6) Construct a Borel set $A \subset \mathbb{R}$ such that

$$0 < \mathcal{L}^1(A \cap I) < \mathcal{L}^1(I)$$

for every non-empty segment I. Is it possible to have $\mathcal{L}^1(A) < \infty$ for such a set?

(7) Let E be a subset of $\mathbb{R}$ with positive Lebsegue measure. Prove that for each $\alpha \in (0,1)$, there exists an open interval I so that $\mathcal{L}^1(E \cap I) \geq \alpha \mathcal{L}^1(I)$. It shows that E contains almost a whole interval. Hint: Choose an open G containing E such that $\mathcal{L}^1(E) \geq \alpha \mathcal{L}^1(G)$ and note that G can be decomposed into disjoint union of open intervals. One of these intervals satisfies our requirement.

(8) Let E be a measurable set in $\mathbb{R}$ with respect to $\mathcal{L}^1$ and $\mathcal{L}^1(E) > 0$. Show that $E - E$ contains an interval $(-a, a)$, $a > 0$. Hint:

(a) U, V open, with finite measure, $x \mapsto \mathcal{L}^1((x + U) \cap V)$ is continuous on $\mathbb{R}$.

(b) A, B measurable, $\mu(A), \mu(B) < \infty$, then $x \mapsto \mathcal{L}^1((x + A) \cap B)$ is continuous. For $A \subset U$, $B \subset V$, try

$$\mathcal{L}^1((x + U) \cap V) - \mathcal{L}^1((x + A) \cap B)| \leq \mathcal{L}^1(U \setminus A) + \mathcal{L}^1(V \subset B).$$

(c) Finally, $x \mapsto \mathcal{L}^1((x + E) \cap E)$ is positive at 0 and if $(x + E) \cap E \neq \phi$, then $x \in E \setminus E$.

(9) Give an example of a continuous map ϕ and a measurable f such that $f \circ \phi$ is not measurable. Hint: May use the function $h = x + g(x)$ where g is the Cantor function as ϕ.