MATH5011 Exercise 4

Standard notations are in force. Problem 4 is for math-majors only.

(1) Identify the Riesz measures corresponding to the following positive functionals ($X = \mathbb{R}$):

(a) $\Lambda_1 f = \displaystyle\int_a^b f\,dx$, and

(b) $\Lambda_2 f = f(0)$.

(2) Let c be the counting measure on $\mathbb{R}$,

$$c(A) = \begin{cases} \#A, & A \neq \phi, \\ 0, & A = \phi. \end{cases}$$

Is there a positive functional

$$\Lambda f = \int f\,dc \quad ?$$

(3) Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, \dots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\,dy.$$

Let μ be the measure associated with this Λ by Theorem 2.14 in [R]. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

(4) Let λ be a Borel measure and μ a Riesz measure on $\mathbb{R}^n$ such that $\lambda(G) = \mu(G)$ for all open sets G. Show that λ coincides with μ on $\mathcal{B}$.

(5) Let μ be a Borel measure on $\mathbb{R}^n$ such that $\mu(K) < \infty$ for all compact K. Show that μ is the restriction of some Riesz measure on $\mathcal{B}$. Hint: Use Riesz representation theorem and Problem 5. This exercise gives a characterization of the Riesz measure on $\mathbb{R}^n$.

(6) Let μ be a Riesz measure on $\mathbb{R}^n$. Show that for every measurable function f, there exists a sequence of continuous function $\{f_n\}$ such that $f_n \to f$ almost everywhere.

(7) A *step function* on $\mathbb{R}$ is a simple function s where $s^{-1}(a)$ is either empty or an interval for every $a \in \mathbb{R}$. Show that for every Lebesgue integrable function f on $\mathbb{R}$, there exists a sequence of step functions $\{s_j\}$ such that

$$\lim_{j \to \infty} \int \big| s_j(x) - f(x) \big| d\mathcal{L}^1(x) = 0.$$

Hint: Approximate f by simple functions (see Ex 2) and then apply Lusin's theorem.