Chapter 4

The Lebesgue Spaces

In this chapter we study L^p-integrable functions as a function space. Knowledge on functional analysis required for our study is briefly reviewed in the first two sections. In Section 1 the notions of normed and inner product spaces and their properties such as completeness, separability, the Heine-Borel property and especially the so-called projection property are discussed. Section 2 is concerned with bounded linear functionals and the dual space of a normed space. The L^p-space is introduced in Section 3, where its completeness and various density assertions by simple or continuous functions are covered. The dual space of the L^p-space is determined in Section 4 where the key notion of uniform convexity is introduced and established for the L^p-spaces. Finally, we study strong and weak convergence of L^p-sequences respectively in Sections 5 and 6. Both are important for applications.

4.1 Normed Spaces

In this and the next section we review essential elements of functional analysis that are relevant to our study of the L^p-spaces. You may look up any book on functional analysis or my notes on this subject attached in this webpage.

Let X be a vector space over $\mathbb{R}$. A norm on X is a map from $X \to [0, \infty)$ satisfying the following three "axioms": For $\forall x, y, z \in X$,

(i) $\|x\| \geq 0$ and is equal to 0 if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|, \qquad \forall \alpha \in \mathbb{R};$ and

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a normed vector space or normed space for short. The norm induces a metric on X given by

$$d(x, y) = \|x - y\| \, .$$

Hence, a normed space is always a metric space and a topological one. We can talk about convergence, continuity, etc. in a normed space. The notion of a Cauchy sequence also makes sense in a metric space, that is, a sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence if for every $\varepsilon > 0$, there is an n_0 such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq n_0$. Recall that a metric is complete if every Cauchy sequence is convergent. A complete normed space is called a Banach space. A general result asserts that every normed space is a dense subspace of some Banach space, and this Banach space is called the completion of the normed space.

Here are some examples of Banach spaces.

- $\mathbb{R}^n$ is a Banach space under the Euclidean norm $\|x\| = \sqrt{\sum x_j^2}$.
- $\ell^p = \left\{x = (x_1, x_2, \dots) : \sum_j |x_j|^p < \infty,\ x_j \in \mathbb{R}\right\}$, $1 \leq p < \infty$, under the ℓ^p-norm: $\|x\|_p = \left(\sum_j |x_j|^p\right)^{1/p}$.
- $C([0,1]) = \{\text{continuous functions on } [0,1]\}$ under the uniform norm (or sup-norm)

$$\begin{aligned}\|f\|_\infty &= \sup\{|f(x)| : x \in [0,1]\} \\ &= \max\{|f(x)| : x \in [0,1]\}.\end{aligned}$$

- $C(K)$ where K is a compact subset in $\mathbb{R}^n$ or a compact metric space. It is also a normed space under the sup-norm. It becomes $C([0,1])$ when $K = [0,1]$.

Let X_1 be a subspace of the normed space $(X, \|\cdot\|)$. Then $(X_1, \|\cdot\|)$ is again a normed space. The followings are some subspaces in $C([0,1])$:

$$\begin{aligned}P([0,1]) &= \{\text{The restrictions of all polynomials on } [0,1]\}, \\ C^1([0,1]) &= \{\text{All continuously differentiable functions on } [0,1]\}, \\ C_0([0,1]) &= \{\text{All continuous functions with } f(0) = f(1) = 0\}, \\ C_c((0,1)) &= \{\text{All continuous functions on } (0,1) \text{ vanishing near } 0 \text{ and } 1\}.\end{aligned}$$

Among these four subspaces, only the third one is a Banach space.

The properties of finite dimensional and infinite dimensional normed spaces are very different. Let us look at the finite dimensional one first.

The Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ is a typical finite dimensional normed space. It has the following four basic properties. Later we will contrast them with those in infinite dimensional spaces.

- *Completeness.* Every finite dimensional normed space is a Banach space.

- *Separability.* Recall that a topological space is separable if it has a countable, dense subset. $\mathbb{R}^n$ is separable as it possesses the countable dense subset $\mathbb{Q}^n$.

- The *Heine-Borel Property.* A normed space has the Heine-Borel property if every bounded sequence in X has a convergent subsequence. It is well-known that $\mathbb{R}^n$ has this property.

- *Projection Property.* Let X_1 be a proper, closed subspace of the normed space X and x_0 not in X_1. X is called to have the projection property if there exists some $z \in X_1$ such that

  $$\|z - x_0\| = \inf \{\|x - x_0\| : x \in X_1\}.$$

  In other words, there is a point on X_1 realizing the distance from x_0 to X_1. $\mathbb{R}^n$ has this property.

In fact, it is instructive to see how this is proved. Let $\{x_j\}$ be a minimizing sequence of d, i.e., $\|x_j - x_0\| \to d$ as $j \to \infty$. Taking $\varepsilon = 1$, there is some j_0 such that $\|x_j - x_0\| \le d+1$, $\forall j \ge j_0$. It follows that $\{x_j\}$ is a bounded sequence: $\|x_j\| \le \|x_j - x_0\| + \|x_0\| \le d + 1 + \|x_0\|$. By the Heine-Borel property, there is a convergent subsequence $\{x_{j_k}\}$, $x_{j_k} \to z$ for some $z \in X_1$. By the continuity of the norm, $\|x_{j_k} - x_0\| \to \|z - x_0\|$. On the other hand, $\|x_{j_k} - x_0\| \to d$ as $\{x_j\}$ is minimizing. We conclude that $\|z - x_0\| = d$.

An inner product on the normed space X over $\mathbb{R}$ is a map : $X \times X \to \mathbb{R}$ satisfying, for all $x, y, z \in X$,

(i) $\langle x, x\rangle \ge 0$, and equal to 0 if and only if $x = 0$,

(ii) $\langle x, y\rangle = \langle y, x\rangle$,

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$, for all $\alpha, \beta \in \mathbb{R}$,

Note it follows that

$$\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle .$$

The pair $(X, \langle\ ,\ \rangle)$ is called in inner product space. The Euclidean product, or the dot product, on $\mathbb{R}^n$ is given by

$$\langle x, y\rangle_e = \sum_{j=1}^{n} x_j y_j.$$

It makes $\mathbb{R}^n$ an inner product space. In general, an inner product induces a norm by

$$\|x\| = \sqrt{\langle x, x\rangle_e}.$$

For instance, the Euclidean norm comes from the Euclidean product. An inner product space is called a Hilbert space if it is a Banach space in the induced norm.

Using the inner product structure on the Euclidean space, we have the following characterization of the point that minimizes the distance between x_0 and X_1 in the projection property, namely, z is the projection of x_0 onto X_1 if and only it satisfies

$$\langle x, x_0 - z \rangle = 0, \quad \forall x \in X_1. \tag{4.1}$$

Now I compare $C([0,1])$, or $C[0,1]$ for simplicity, with the Euclidean space. It is easy to show that $C[0,1]$ is an infinite dimensional vector space.

- *Completeness.* Yes, $C[0,1]$ is complete under the sup-norm. In elementary analysis we proved that the uniform limit of a sequence of continuous functions is continuous.

- *Separability.* Yes. By the Weierstrass approximation theorem, every continuous function on $[0,1]$ can be approximated by polynomials with rational coefficients. Thus the collection of all these polynomials forms a countable, dense set in $C([0,1])$. More generally, it can be shown that $C(K)$ is separable when K is a compact metric space.

- The *Heine-Borel Property.* Not any more. In fact, let f_n be the piecewise linear function which is equal to 0 at 0 and $[1/n,1]$ and $f_n(1/2n) = 1$. Clearly $f_n(x) \to 0 \; \forall x \in [0,1]$. However, it does not contain any uniformly convergent subsequence. Suppose on the contrary that there exists some $\{f_{n_j}\}$ converging to some continuous f. Then f must be the zero function. For every $\varepsilon > 0$, there is some j_0 such that

  $$\left\|f_{n_j} - 0\right\|_\infty < \varepsilon, \quad j \geq j_0.$$

  Taking $\varepsilon = 1/2$, $\left\|f_{n_j}\right\|_\infty < 1/2$ for all large j. But $\left\|f_{n_j}\right\|_\infty = 1$, contradiction holds. Hence the Heine-Borel property no longer holds in $C([0,1])$. It is a bit striking that this property characterize finite dimensionality.

  Theorem 4.1. *The Heine-Borel property holds on a normed space X if and only if X is finite dimensional.*

  See, for instance, Theorem 2.1.2 in my notes on functional analysis.

- *Projection Property.* No. But the construction of an example is not easy. I just want to point out that very often a property so obvious for a finite dimensional space may not hold in an infinite dimensional space. If it holds, more effort is need to established it. Fortunately, this property holds in Hilbert spaces.

Theorem 4.2. *Let X be a Hilbert space and $x_0 \notin X_1$ where X_1 is a proper closed subspace. There exists a unique $z \in X_1$ such that*

$$\|z - x_0\| = \inf \{\|x - x_0\| : x \in X_1\},$$

and (4.1) holds.

Proof. First of all, by expanding $\|x \pm y\|^2 = \langle x \pm y, x \pm y\rangle$, we have the parallelogram rule

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

which is true on every inner product space. Now, let $\{y_k\}$ be a minimizing sequence for $d = \inf \{\|y - x_0\| : y \in X_1\}$, that is, $\|y_k - x_0\| \to d$ as $k \to \infty$. Using the parallelogram rule, we have

$$\|y_k - y_m\|^2 = 2(\|y_k - x_0\|^2 + \|y_m - x_0\|^2) - 2\|\frac{y_k + y_m}{2} - x_0\|^2.$$

As $(y_k + y_m)/2$ belongs to X_1, $\|(y_k + y_m)/2 - x_0\| \geq d$. Given $\varepsilon > 0$, there exists some n_0 such that $\|y_n - x_0\|^2 - d^2 < \varepsilon$ for all $n \geq n_0$. It follows that

$$\|y_k - y_m\| \leq 2d^2 + 2\varepsilon - 2d^2 = 2\varepsilon,$$

for all $k, m \geq n_0$. Thus $\{y_k\}$ is a Cauchy sequence in X_1. As X_1 is closed, this sequence converges to some z in X_1 which satisfies $\|z - x_0\| = d$.

In case there is another $z_1 \in X_1$ satisfying $\|z_1 - x_0\| = d$, we have

$$\begin{aligned} \|z_1 - z\|^2 &= 2(\|z_1 - x_0\|^2 + \|z - x_0\|^2 - 2\|\frac{z_1 + z}{2} - x_0\|^2 \\ &= 2d^2 - 2\|\frac{z_1 + z}{2} - x_0\|^2 \leq 0, \end{aligned}$$

which forces $z_1 = z$. □

4.2 Bounded Linear Functionals

A linear functional on a vector space is a linear map from X to $\mathbb{R}$. It is called *bounded* if there is some M such that

$$|\Lambda x| \leq M \|x\|, \quad \forall x \in X. \tag{4.2}$$

It means Λ maps bounded sets in X to bounded intervals.

Proposition 4.3. *A linear functional on the normed space X is bounded if and only if it is continuous.*

All bounded linear functionals on X form a normed vector space under the *operator norm*:

$$\|\Lambda\|_{op} = \sup\{|\Lambda x| : \|x\| \le 1\}.$$

Furthermore, it is complete regardless of the completeness of X. This Banach space is called the dual space of X and is denoted by $(X', \|\cdot\|_{op})$.

The dual space of $\mathbb{R}^n$ can be identified with itself.

Theorem 4.4. *There is a bijective, norm-preserving linear map between* $((\mathbb{R}^n)', \|\cdot\|_{op})$ *and* $(\mathbb{R}^n, \|\cdot\|)$.

In view of this, the Euclidean space is self-dual. Indeed, let $\Lambda \in (\mathbb{R}^n)'$. For $x \in \mathbb{R}^n$, $x = \sum_j \alpha_j e_j$ where $\{e_j\}$ is the canonical basis of $\mathbb{R}^n$. We have

$$\Lambda x = \Lambda\left(\sum \alpha_j e_j\right) = \sum \alpha_j \Lambda(e_j).$$

The map $\Phi : (\mathbb{R}^n)' \to \mathbb{R}^n$ given by $\Phi(\Lambda) = (\Lambda e_1, \dots, \Lambda e_n)$ is the desired bijective, norm-preserving linear map.

A fundamental result concerning linear functionals is the Hahn-Banach theorem.

Theorem 4.5. *Let X_1 be a subspace of the vector space X. Suppose p is a sublinear function defined on X. Any linear functional Λ on X_1 satisfying $|\Lambda x| \le p(x)$, $\forall x \in X_1$, has an extension to a linear functional Λ' on X satisfying $|\Lambda' x| \le p(x)$, $\forall x \in X$.*

A function $p : X \to [0, \infty)$ is sublinear if for all $x, y \in X$, $\alpha \ge 0$,

$$p(x+y) \le p(x) + p(y), \quad \text{and}$$
$$p(\alpha x) = \alpha p(x), \ \alpha \ge 0$$

hold. The norm is of course a sublinear function on X. Taking the sublinear function to be a constant multiple of the norm, we have the following version of Hahn-Banach theorem that applies to normed spaces.

Theorem 4.6. *Let X_1 be a subspace of the normed space X. Every bounded linear functional Λ on X_1 satisfying $|\Lambda x| \le M\|x\|$, $\forall x \in X_1$, has an extension to a bounded linear functional Λ' on X satisfying $|\Lambda' x| \le M\|x\|$.*

We conclude this section by a discussion on the "codimension one" property of the kernel of Λ.

It is easy to see that $\ker \Lambda = \{x : \Lambda x = 0\}$ is a closed subspace of the normed space X whenever $\Lambda \in X'$.

Proposition 4.7. *Let $\Lambda \neq 0$ be a non-zero linear functional on X and $x_1 \notin \ker \Lambda$. For each $x \in X$, there exist $\alpha \in \mathbb{R}$ and $x_2 \in \ker \Lambda$ such that*

$$x = \alpha x_1 + x_2.$$

Proof. Let $x_2 = x - \alpha x_1$ where $\alpha = \Lambda x / \Lambda x_1$. Then

$$\begin{aligned} \Lambda x_2 &= \Lambda(x - \alpha x_1) \\ &= \Lambda x - \alpha \Lambda x_1 \\ &= \Lambda x - \frac{\Lambda x}{\Lambda x_1} \Lambda x_1 = 0. \end{aligned}$$

□

This proposition tells us that the space is spanned by the kernel of a non-zero bounded linear functional together with a one dimensional subspace spanned any fixed vector lying outside the kernel. Consequently, the kernel is of "codimension one".

Corollary 4.8. *Let Λ_1 and Λ_2 be two linear functionals on X such that* $\ker \Lambda_1 = \ker \Lambda_2$. *Then $\Lambda_2 = c\Lambda_1$ for some non-zero constant c.*

Proof. Pick $x_0 \notin \ker \Lambda_1$. From $x = (\Lambda_1 x / \Lambda_1 x_0) x_0 + y, y \in \ker \Lambda_1 = \ker \Lambda_2$, we have $\Lambda_2 x = (\Lambda_1 x / \Lambda_1 x_0) \Lambda_2 x_0$, hence $\Lambda_2 x = \alpha \Lambda_1 x$ where $\alpha = \Lambda_2 x_0 / \Lambda_1 x_0$. □

The codimension one property has the following interesting consequence for Hilbert spaces.

Theorem 4.9. *Let $\Lambda \in X'$ where X is a Hilbert space. There exists a unique $w \in X$ such that $\Lambda x = \langle x, w \rangle$, $\forall x \in X$.*

With further effort, one can show that $\Lambda \mapsto w$ is a bijective, norm-preserving linear map. Generalizing Theorem 4.4, every Hilbert space is self-dual.

Proof. Assume that the functional Λ is non-zero. We pick $x_0 \notin \ker \Lambda$ and let $z \in \ker \Lambda$ satisfy $\langle x, x_0 - z \rangle = 0$ for all $x \in \ker \Lambda$. Therefore, the linear functional Λ' given by $\Lambda' x = \langle x, x_0 - z \rangle$ has the same kernel as Λ and so these two functionals differ by a multiplying constant. In fact, from the last corollary we have

$$\begin{aligned} \Lambda' x &= \frac{\Lambda' x_0}{\Lambda x_0} \Lambda x \\ &= \frac{\langle x_0, x_0 - z \rangle}{\Lambda x_0} \Lambda x \\ &= \frac{\|x_0 - z\|^2}{\Lambda x_0} \Lambda x, \end{aligned}$$

which implies $\Lambda x = \langle x, w \rangle$ where

$$w = \frac{(\Lambda x_0)(x_0 - z)}{\|x_0 - z\|^2}.$$

□

4.3 Lebesgue Spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 < p < \infty$. A measurable function f is called a *p-integrable function* if $|f|^p$ is integrable.

Proposition 4.10 (**Hölder's Inequality**). *Let f and g be measurable. For $1 < p < \infty$,*

$$\int_X |fg| \, d\mu \le \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \left(\int_X |g|^q \, d\mu \right)^{\frac{1}{q}},$$

where q is conjugate to p. Moreover, when the right hand side of this inequality is finite, equality sign holds if and only if either (a) f or g vanishes a.e. or (b) there are some $\alpha, \beta \ge 0$, $\alpha\beta \neq 0$, such that $\alpha |f|^p = \beta |g|^q$ a.e. .

Recall that for $p, q \in [1, \infty]$, p and q are conjugate if $1/p + 1/q = 1$.

Proposition 4.11 (**Minkowski's Inequality**). *Let f and g be measurable. For $1 \le p < \infty$,*

$$\left(\int_X |f + g|^p \, d\mu \right)^{\frac{1}{p}} \le \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu \right)^{\frac{1}{p}}.$$

Minkowski's inequality could be deduced from Hölder's inequality, and Hölder's inequality follows from Young's inequality or Jensen's inequality. You are referred to [R] for the proofs of these inequalities.

Let

$$L^p(X, \mathcal{M}, \mu) = \{\text{all } p\text{-integrable functions on } (X, \mathcal{M}, \mu)\}.$$

Letting

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

it follows from Minkowski's inequality that

$$\|f + g\|_p \le \|f\|_p + \|g\|_p, \qquad f, g \in L^p(X, \mathcal{M}, \mu).$$

It is almost a norm except that $\|f\|_p = 0$ implies $f = 0$ almost everywhere but not necessarily everywhere. To obtain a normed space, we introduce an equivalence relation $\sim$ on $L^p(X, \mathcal{M}, \mu)$ by setting

$$f \sim g \text{ if and only if } f - g = 0 \text{ a.e.}$$

This equivalence relation partitions all L^p-functions into equivalent classes $\widetilde{f}$. Define

$$\widetilde{\|\widetilde{f}\|}_p = \|f\|_p, \quad f \in \widetilde{f}.$$

Then $(\widetilde{L^p}(X, \mathcal{M}, \mu), \widetilde{\|\cdot\|}_p)$ becomes a normed space. In practise, people do not distinguish $\widetilde{\|\cdot\|}_p$ from $\|\cdot\|_p$. They simply write $L^p(X, \mathcal{M}, \mu)$ or even $L^p(\mu)$ for $(\widetilde{L}^p(X, \mathcal{M}, \mu), \widetilde{\|\cdot\|}_p)$.

Theorem 4.12. *Let $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$. There exists $f \in L^p(\mu)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. Consequently, $\widetilde{L}^p(\mu)$ is a Banach space for $1 \le p < \infty$.*

Proof. Let $\{f_n\}$ be a Cauchy sequence of p-integrable functions. For $\varepsilon = 1/2^j$, there exists n_j such that $\|f_n - f_m\|_p < 1/2^j$ $\forall n, m \ge n_j$. We could choose n_j such that $n_j \uparrow \infty$. Then

$$\left\|f_{n_{j+1}} - f_{n_j}\right\|_p < \frac{1}{2^j}, \qquad \forall j \ge 1.$$

Pick a null set N_1 so that all f_{n_j} are finite in $X_1 = X \setminus N_1$. Set

$$g = \sum_{j=1}^{\infty} \left|f_{n_{j+1}} - f_{n_j}\right|$$

on X_1 and $g = 0$ in N_1. We have

$$
\begin{aligned}
\int_X g^p \, d\mu &= \int_X \Big(\sum_{j=1}^{\infty} \big| f_{n_{j+1}} - f_{n_j} \big| \Big)^p \, d\mu \\
&= \int_X \Big(\lim_{k\to\infty} \sum_{j=1}^{k} \big| f_{n_{j+1}} - f_{n_j} \big| \Big)^p \, d\mu \\
&= \int_X \lim_{k\to\infty} \Big(\sum_{j=1}^{k} \big| f_{n_{j+1}} - f_{n_j} \big| \Big)^p \, d\mu \\
&\quad \text{(continuity of } z \mapsto z^p\text{)} \\
&= \lim_{k\to\infty} \int_X \Big(\sum_{j=1}^{k} \big| f_{n_{j+1}} - f_{n_j} \big| \Big)^p \, d\mu \\
&\quad \text{(monotone convergence theorem)} \\
&= \varliminf_{k\to\infty} \int_X \Big(\sum_{j=1}^{k} \big| f_{n_{j+1}} - f_{n_j} \big| \Big)^p \, d\mu \\
&\le \varliminf_{k\to\infty} \Big[\sum_{j=1}^{k} \Big(\int_X \big| f_{n_{j+1}} - f_{n_j} \big|^p \, d\mu \Big)^{\frac{1}{p}} \Big]^p \\
&\quad \text{(Minkowski inequality)} \\
&\le \varliminf_{k\to\infty} \Big(\sum_{j=1}^{\infty} \frac{1}{2^j} \Big)^p \\
&\le 1.
\end{aligned}
$$

Hence $g \in L^p(\mu)$ and in particular is finite almost everywhere. Let $N_2 \equiv \{x \in X_1 : g(x) = \infty\}$ and $X_2 = X_1 \setminus N_2 = X \setminus (N_1 \cup N_2)$ and set

$$
h_k = \sum_{j=1}^{k} \left(f_{n_{j+1}} - f_{n_j} \right).
$$

For $x \in X_2$,

$$
|h_k(x) - h_l(x)| \le \sum_{j=l+1}^{k} |f_{n_{j+1}} - f_{n_j}|(x) \to 0, \quad \text{as } k, l \to \infty,
$$

since g is finite in X_2. We conclude that $\{h_k(x)\}$ is a numerical Cauchy sequence and hence

$$
h = \sum_{j=1}^{\infty} \left(f_{n_{j+1}} - f_{n_j} \right)
$$

is well-defined in X_2. We may set $h(x) = 0$ elsewhere so that it is a measurable function defined in X. From $|h_k| \leq g$ we also know that $h \in L^p(X)$ as a result of Lebesgue's dominiated convergence theorem. By the same reason, using $|h - h_k| \leq |h| + |h_k| \leq 2g$, we conclude $\|h - h_k\|_p \to 0$ as $k \to \infty$. It follows that $\|f - f_{n_{k+1}}\|_p = \|h - h_k\|_p \to 0$ as $k \to \infty$. (In fact, we also have $f_{n_{k+1}}(x) - f(x) = h_k(x) - h(x) \to 0$ at every $x \in X_2$, that is, the subsequence $\{f_{n_k}\}$ converges to f almost everywhere.) We conclude that $f_{n_k} \to f$ in L^p-norm and almost everywhere as $n_k \to \infty$.

Now, for $\varepsilon > 0$, there is some n_0 such that

$$\|f_n - f_m\|_p < \frac{\varepsilon}{2}, \quad \forall n, m \geq n_0.$$

On the other hand, there is some k_0 such that

$$\|f_{n_k} - f\|_p < \frac{\varepsilon}{2}, \quad \forall k \geq k_0.$$

We pick $n_k \geq n_0$ and $k \geq k_0$, then

$$\|f_n - f\|_p \leq \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whence $\{f_n\}$ converges to f in $L^p(\mu)$.

Finally, for a Cauchy sequence $\{\widetilde{f_n}\}$ in $\widetilde{L}^p(\mu)$, pick $f_n \in \widetilde{f_n}$. Then $\{f_n\}$ is a Cauchy sequence of L^p-integrable functions. From what we have just proved, there exists a L^p-integrable function f such that $\|f_n - f\|_p \to 0$. Then $\|\widetilde{f_n} - \widetilde{f}\|_p = \|f_n - f\|_p \to 0$ once we take $\widetilde{f}$ to be the equivalence class containing f. □

From the above proof we also have the following useful information.

Corollary 4.13. *If $f_n \to f$ in $L^p(\mu)$, there exists a subsequence $\{f_{n_k}\}$ converging to f a.e. .*

This result is already known for $p = 1$, see the last section in Chapter 1.

After ascertaining that $L^p(\mu)$ is a Banach space, we study the approximation properties of these spaces.

First, we consider the density of simple functions in $L^p(\mu)$. Recall that a simple function s is in the form

$$s(x) = \sum_{j=1}^{n} \alpha_j \chi_{E_j}(x), \quad \alpha_j \neq 0 \in \mathbb{R},\ E_j \text{ measurable.}$$

Let

$$\mathcal{S} = \{s : s \text{ is a simple function with } \mu(E_j) < \infty,\ \forall j = 1, \ldots, n\}.$$

Proposition 4.14. *The set $\{s_n\} \subset \mathcal{S}$ is dense in $L^p(\mu)$ for $p \in [1, \infty)$.*

Proof. By writing $f = f^+ - f^-$ and noting $|f|^p = |f^+|^p + |f^-|^p$, it suffices to prove the proposition for non-negative f. By Theorem 1.6, there exists a sequence of simple functions $\{s_k\} \uparrow f$. Using $0 \leq f - s_k \leq f$, we can apply the Lebesgue's dominated convergence theorem to get

$$\lim_{k\to\infty} \int |f - s_k|^p, d\mu = \int \lim_{k\to\infty} |f - s_k|^p, d\mu = 0.$$

It remains to verify $s_k \in \mathcal{S}, n \geq 1$, but this is evident. As $s_k = \sum_{j=1}^{n} \alpha_j \chi_{E_j}$ where $\alpha_j > 0$. As $s_k \leq f$, $s_k^p \leq |f|^p$ implies $\alpha_j^p \mu(E_j) \leq \int |f|^p \, d\mu < \infty$ for each j. So $\mu(E_j) < \infty$ and $s_k \in \mathcal{S}$ for all k. $\square$

When the underlying space X has a topological structure, one should study the density of continuous functions. We have

Proposition 4.15. *Let X be a locally compact Hausdorff space and μ a Riesz measure on X. Then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$.*

Proof. Using Proposition 4.13 and the definition of $\mathcal{S}$, it suffices to show that for every measurable E, $\mu(E) < \infty$, we can find a sequence of continuous functions $\{\varphi_j\} \subset C_c(X)$ such that $\varphi_j \to \chi_E$ in L^p-norm. By the outer and inner regularity of E, there are descending open sets $\{G_j\} \downarrow E$ and ascending compact sets $\{K_j\} \uparrow E$ with $\mu(G_1) < \infty$. As X is locally compact Hausdorff, by Urysohn's lemma, for each j there is some $\varphi_j \in C_c(X)$, $\varphi_j \equiv 1$ on K_j, $\mathrm{spt}\varphi_j \subset G_j$, $0 \leq \varphi_j \leq 1$ on X. Using $\varphi_j \leq \chi_{G_1}$ and $\mu(G_1) < \infty$, one can apply the Lebesgue's dominated convergencr theorem to conclude $\varphi_j \to \chi_E$ in $L^p(\mu)$. $\square$

Proposition 4.16. *The space $L^p(\mathbb{R}^n)$ is separable for $1 \leq p < \infty$.*

Here $L^p(\mathbb{R}^n)$ stands for the L^p-space with respect to $\mathcal{L}^n$, the n-dimensional Lebesgue measure.

Proof. Let $\mathcal{P}_n$ be the collection of the restrictions of all polynomials with rational coefficients on the ball B_n centered at the origin (setting them to be zero outside the ball) and and $\mathcal{P} = \bigcup_n \mathcal{P}_n$. Then $\mathcal{P}$ is a countable subset in $L^p(\mathbb{R}^n)$. For $f \in L^p(\mathbb{R}^n)$, given $\varepsilon > 0$, there exists some $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \varepsilon/2$. Let n be large so that $\mathrm{spt} g \subset B_n$. By Weierstrass approximation theorem, there is a polynomial $p \in \mathcal{P}_n$ such that $\|g - p\|_\infty < \varepsilon/2 \, |\mathcal{L}^n(B_n)|^{1/q}$. Using

$$\|g - p\|_p = \left(\int |g - p|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \leq |\mathcal{L}^n(B_n)|^{\frac{1}{q}} \, \|g - p\|_\infty ,$$

we have

$$
\begin{aligned}
\|f-p\|_p &\leq \|f-g\|_p + \|g-p\|_p \\
&< \frac{\varepsilon}{2} + |\mathcal{L}^n(B_n)|^{\frac{1}{q}} \frac{\varepsilon}{2\,|\mathcal{L}^n(B_n)|^{\frac{1}{q}}} \\
&= \varepsilon.
\end{aligned}
$$

$\square$

4.4 The Dual Space of $L^p(\mu)$

We would like to determine the dual space of $L^p(\mu)$ for $p \in (1,\infty)$. Let $g \in L^q(\mu)$ where q is conjugate to p. We define

$$
\Lambda_g f = \int gf \, d\mu, \quad f \in L^p(\mu).
$$

By Hölder inequality,

$$
\begin{aligned}
|\Lambda_g f| &\leq \left| \int gf \, d\mu \right| \\
&\leq \int |gf| \, d\mu \\
&\leq \|g\|_q \, \|f\|_p \,, \qquad \forall f \in L^p(\mu).
\end{aligned}
$$

Hence, $\Lambda_g \in L^p(\mu)'$ and

$$
\|\Lambda_g\| \leq \|g\|_q
$$

holds. On the other hand, the function $f_1 = |g|^{q-2}\, g$ satisfies

$$
\int |f_1|^p \, d\mu = \int |g|^{(q-1)p} \, d\mu = \int |g|^q \, d\mu < \infty,
$$

so $f_1 \in L^p(\mu)$. We have

$$
|\Lambda_g f_1| \leq \|\Lambda_g\| \, \|f_1\|_p \,,
$$

that is,

$$
\int |g|^q \, d\mu \leq \|\Lambda_g\| \left(\int |g|^q \, d\mu \right)^{\frac{1}{p}},
$$

which means

$$
\|g\|_q \leq \|\Lambda_g\| \,.
$$

We conclude that the linear map $\Phi : L^q(\mu) \to L^p(\mu)'$ given by $g \mapsto \Lambda_g$ is norm-preserving. Note that norm-preserving implies that Φ is injective. We will show that it is also surjective. We have

Theorem 4.17. *Let $\Lambda \in L^p(\mu)', 1 < p < \infty$. There is a unique $g \in L^q(\mu)$ such that $\Lambda = \Lambda_g$. The correspondence between Λ and g sets up a norm-preserving linear bijection between the dual space of $\widetilde{L}^p(\mu)$ and $\widetilde{L}^q(\mu)$.*

To prepare for the proof of this theorem, we introduce a notion of uniform convexity in functional analysis. A normed space is called *uniformly convex* if for any two unit vectors satisfying $\|x - y\| \geq \varepsilon$ for some $\varepsilon \in (0, 1)$, there exists some $\theta \in (0, 1)$ depending on ε only such that $\|(x + y)/2\| \leq 1 - \theta$. This terminology comes from the shape of the unit sphere. For instance, the plane $\mathbb{R}^2$ in the Euclidean metric is uniformly convex, but is not in the maximal norm $\|(x, y)\|_\infty = \max\{|x|, |y|\}$, as its unit sphere is given by the boundary of the unit square $[-1, 1]^2$. Taking $(1, 1)$ and $(1, -1)$ on this unit sphere, $\|((1, 1) + (1, -1))/2\|_\infty = 1$. Every Hilbert space is uniformly convex, for, by the parallelogram law, for any two units vectors x and y,

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) = 4,$$

thus,

$$\left\|\frac{x + y}{2}\right\|^2 = 1 - \frac{1}{4}\|x - y\|^2 \leq 1 - \frac{\varepsilon^2}{4} < 1.$$

So, we can find some $\theta \in (0, 1)$ such that $\|(x + y)/2\| < 1 - \theta$. We will use uniform convexity in the following way: In case $\|(x + y)/2\| \to 1$, then $\|x - y\| \to 0$.

The projection property holds in a uniformly convex space. This is the crucial property that we will use.

Proposition 4.18. *Let X_1 be a proper, closed subspace of a uniformly convex Banach space X and x_0 a point lying outside X_1. There exists a unique $z \in X_1$ such that*

$$\|z - x_0\| \leq \|y - x_0\|, \qquad \forall y \in X_1.$$

Proof. Let $\{y_n\} \in X_1$ be minimizing the distance from x_0 to X_1, that is, $\|y_n - x_0\| \to d$ as $n \to \infty$. Noting that the distance d is positive and $\{y_n\}$ is bounded, we have

$$\begin{aligned}
& \liminf_{n\to\infty}\left\|\frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|}\right\| \\
\geq\ & \liminf_{n\to\infty}\left|\left\|\frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} - \left(\frac{y_n - x_0}{d} + \frac{y_m - x_0}{d}\right)\right\| - \left\|\frac{y_n - x_0}{d} + \frac{y_m - x_0}{d}\right\|\right| \\
\geq\ & \frac{2}{d}\liminf_{n\to\infty}\left\|\frac{1}{2}(y_n + y_m)\right\| \\
\geq\ & 2,
\end{aligned}$$

after using the fact that $(y_n + y_m)/2 \in X_1$ and $\|(y_n + y_m)/2\| \geq d$. In other words,

we have

$$\liminf_{n\to\infty} \left\| \frac{1}{2}\Big(\frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \Big) \right\| \geq 1.$$

On the other hand, clearly we have

$$\left\| \frac{1}{2}\Big(\frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \Big) \right\| \leq 1,$$

so

$$\lim_{n,m\to\infty} \left\| \frac{1}{2} \left(\frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \right) \right\| = 1 \ .$$

By uniform convexity,

$$\begin{aligned} \lim_{n,m\to\infty} \|y_n - y_m\| &= d \lim_{n,m\to\infty} \left\| \frac{y_n - x_0}{d} - \frac{y_m - x_0}{d} \right\| \\ &= d \lim_{n,m\to\infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} - \frac{y_m - x_0}{\|y_m - x_0\|} \right\| \\ &\to 0, \end{aligned}$$

as $n, m \to \infty$, that is, $\{y_n\}$ is a Cauchy sequence. By the completeness of X_1, $y_n \to z \in X_1$ for some $z \in X_1$ and $\|z - x_1\| = d$.

If there exists another $z' \in X_1$ satisfying $\|z' - x_0\| = d$, let $\|z - z'\| \geq \varepsilon$ for some $\varepsilon > 0$. For some $\theta \in (0, 1)$ determined by ε,

$$\left\| \frac{1}{2} \frac{z' - x_0}{d} + \frac{1}{2} \frac{z - x_0}{d} \right\| = \left\| \frac{\frac{1}{2}(z + z') - x_0}{d} \right\| < 1 - \theta.$$

That means the distance between x_0 and $(z + z')/2 \in X_1$ is strictly less than d, but this is impossible.

□

Theorem 4.19. *The space $L^p(\mu)$, $1 < p < \infty$, is uniformly convex. Consequently, it satisfies the projection property.*

This theorem relies on two inequalities.

Proposition 4.20 (**Clarkson's Inequalities**).
Let f and g be in $L^p(\mu)$. We have

(a) *For $p \geq 2$,*

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right).$$

(b) *For $p \in (1, 2)$,*

$$\|f + g\|_p^q + \|f - g\|_p^q \leq 2 \left(\|f\|_p^p + \|g\|_p^p \right)^{q-1},$$

where p and q are conjugate.

Now we prove Clarkson's inequalities.

Proof. (a) We note that (a) follows from integrating the following elementary inequality

$$\left|\frac{x+y}{2}\right|^p + \left|\frac{x-y}{2}\right|^p \leq \frac{1}{2}\left(|x|^p + |y|^p\right), \quad p \geq 2,\ x, y \in \mathbb{R},$$

after substituting x and y by $f(x)$ and $g(x)$ respectively. Again, using the symmetry in this inequality, it suffices to show it under $x > y > 0$. Setting $z = y/x$, we further reduce to

$$\left(\frac{1+z}{2}\right)^p + \left(\frac{1-z}{2}\right)^p \leq \frac{1}{2}(1+z^p), \quad z \in [0,1].$$

Let

$$\varphi(z) = \left(\frac{1+z}{2}\right)^p + \left(\frac{1-z}{2}\right)^p - \frac{1}{2}(1+z^p).$$

Then $\varphi(0) = 2^{-p} - 2^{-1} < 0$, $\varphi(1) = 1 - 1 = 0$. If $\varphi'(z) \geq 0$, then $\varphi(z) \leq 0$ on $[0,1]$ and we are done. We compute

$$\begin{aligned}\varphi'(z) &= \frac{p}{2^p}\left[(1+z)^{p-1} - (1-z)^{p-1} - 2^{p-1}z^{p-1}\right] \\ &= \frac{p}{2^p}z^{p-1}\left[(w+1)^{p-1} - (w-1)^{p-1} - 2^{p-1}\right], \quad w = \frac{1}{z} \in [1,\infty).\end{aligned}$$

Let $h(w) = (w+1)^{p-1} - (w-1)^{p-1} - 2^{p-1}$. Then $h(1) = 0$ and $h'(w) \geq 0$ (easily seen after using $p \geq 2$), so $h(w) \geq 0$ and $\varphi'(z) \geq 0$. We have established (a).
(b) We need another inequality

$$|x+y|^q + |x-y|^q \leq 2\left(|x|^p + |y|^p\right)^{q-1}, \quad x, y \in \mathbb{R}. \tag{4.3}$$

As before it reduces to

$$(1+z)^q + (1-z)^q \leq 2(1+z^p)^{q-1}, \quad z \in [0,1].$$

Let

$$f(\alpha, z) = (1+\alpha^{1-q}z)(1+\alpha z)^{q-1} + (1-\alpha^{1-q}z)(1-\alpha z)^{q-1}.$$

Then $f(1,z) = (1+z)^q + (1-z)^q$ and $f(z^{p-1}, z) = 2(1+z^p)^{q-1}$. As $z^{p-1} < 1$, it suffices to show $\partial f/\partial\alpha(\alpha,z)\le 0$ for $z,\alpha\in(0,1)$. We have

$$\begin{aligned}\frac{\partial f}{\partial\alpha}(\alpha,z) &= (1-q)\alpha^{-q}z(1+\alpha z)^{q-1}+(1+\alpha^{1-q}z)(q-1)(1+\alpha z)^{q-2}z\\ &\quad -(1-q)\alpha^{-q}z(1-\alpha z)^{q-1}-(1-\alpha^{1-q}z)(1-\alpha z)z(q-1)\\ &= (1-q)z\left[\alpha^{-q}(1+\alpha z)-1-\alpha^{1-q}z\right](1+\alpha z)^{q-2}\\ &\quad -(1-q)z\left[\alpha^{-q}(1-\alpha z)-1+\alpha^{1-q}z\right](1-\alpha z)^{q-2}\\ &= (1-q)z(\alpha^{-q}-1)\left[(1+\alpha z)^{q-2}-(1-\alpha z)^{q-2}\right]\\ &\le 0.\end{aligned}$$

Using $q>2$ and $\alpha\in(0,1)$, (4.3) holds. However, (b) doesn't come from (4.3) by integration. We need one more result, namely, for $0<p<1$ and non-negative f,g,

$$\|f+g\|_p\ge\|f\|_p+\|g\|_p\,. \tag{4.4}$$

We leave the proof of (4.4) as an exercise. Now,

$$\|f\|_p^q=\left(\int|f|^p\,d\mu\right)^{\frac{q}{p}}=\left(\int|f|^{q(p-1)}\,d\mu\right)^{\frac{1}{p-1}}=\||f|^q\|_{p-1}\,,$$

and

$$\begin{aligned}\|f+g\|_p^q+\|f-g\|_p^q &= \||f+g|^q\|_{p-1}+\||f-g|^q\|_{p-1}\\ &\le \||f+g|^q+|f-g|^q\|_{p-1}\qquad(\text{use } 0<p-1<1 \text{ and } (4.4))\\ &= \left[\int(|f+g|^q+|f-g|^q)^{p-1}\,d\mu\right]^{\frac{1}{p-1}}\\ &\le 2\left[\int(|f|^p+|g|^p)^{(q-1)(p-1)}\,d\mu\right]^{\frac{1}{p-1}}\qquad(\text{by } (4.3))\\ &= 2\left(\int|f|^p\,d\mu+\int|g|^p\,d\mu\right)^{\frac{1}{p-1}}\\ &= 2\left(\|f\|_p^p+\|g\|_p^p\right)^{q-1},\end{aligned}$$

and (b) follows after noting $(p-1)(q-1)=1$.

□

Proof of Theorem 4.16. It remains to show that Φ is onto, i.e., for $\Lambda\in L^p(\mu)'$, there is some $g\in L^q(\mu)$ such that

$$\Lambda f=\int fg\,d\mu,\quad \forall f\in L^p(\mu).$$

We fix some $f_1\in L^p(\mu)\setminus\ker\Lambda$. As $L^p(\mu)$ is uniformly convex, there is an

$h_0 \in \ker \Lambda$ such that

$$\|h_0 - f_1\|_p \le \|f - f_1\|_p \,, \quad \forall f \in \ker \Lambda.$$

That is to say,

$$\varphi(t) = \|h_0 + tf - f_1\|_p^p$$

attains its minimum at $t = 0$ and $\varphi'(0) = 0$ (the differentiability of φ is left as an exercise). We have

$$0 = \varphi'(0) = p \int |h_0 - f_1|^{p-2} (h_0 - f_1) f \, d\mu, \quad \forall f \in \ker \Lambda.$$

Letting $g_1 = |h_0 - f_1|^{p-2} (h_0 - f_1)$, we've

$$\int |g_1|^q \, d\mu = \int |h_0 - f_1|^{(p-1)q} = \int |h_0 - f_1|^p \, d\mu < \infty,$$

so $g_1 \in L^q(\mu)$ and

$$\int g_1 f \, d\mu = 0, \quad \forall f \in \ker \Lambda. \tag{4.5}$$

On the other hand, recall that $h_0 - f_1 \notin \ker \Lambda$, so for every $f \in L^p(\mu)$, by the codimension one property of Λ we can find α and $f_2 \in \ker \Lambda$ such that

$$f = \alpha(h_0 - f_1) + f_2, \quad \alpha = \frac{\Lambda f}{\Lambda(h_0 - f_1)}.$$

Multiply both sides by g_1 and using (4.5) to get

$$\begin{aligned} \int f g_1 \, d\mu &= \alpha \int (h_0 - f_1) g_1 \, d\mu + \int f_2 g_1 \, d\mu \\ &= \alpha \int |h_0 - f_1|^p \, d\mu, \end{aligned}$$

i.e.,

$$\Lambda f = \frac{\Lambda(h_0 - f_1)}{\|h_0 - f_1\|_p^p} \int f g_1 \, d\mu, \quad \forall f \in L^p.$$

So

$$\Lambda f = \int f g \, d\mu$$

where

$$g = \frac{\Lambda(h_0 - f_1)}{\|h_0 - f_1\|_p^p} \, g_1.$$

$\square$

So far we have left out the dual space of $L^1(\mu)$, which is given by $L^\infty(\mu)$. Now we discuss it. A μ-measurable function in $(X, \mathcal{M}, \mu)$ is called *essentially bounded* if there exist some M and a measure zero set N such that $|f(x)| \leq M$, $\forall x \in X \setminus N$. The collection of all essentially bounded functions form a vector space. For an essentially bounded function, let its *essential supremum norm* be

$$\|f\|_\infty = \inf \{\beta : \mu \{x : |f(x)| \geq \beta\} = 0\} .$$

One can show that $(L^\infty(\mu), \|\cdot\|_\infty)$ forms a Banach space after identifying f and g if they are different at a set of measure zero. The space $L^\infty(\mu)$ is not as nice as $L^p(\mu)$ for $p \in (1, \infty)$. For example, in the case of the Lebesgue measure, $L^\infty(\mathbb{R}^n, \mathcal{L}^n)$ is not separable and continuous functions are not dense in it. Nevertheless, it still comes up as the dual space of $L^1(\mu)$ for a σ-finite measure μ including the n-dimensional Lebesgue measure. Duality may not hold when the measure is not σ-finite, see [R]. On the other hand, it is clear that every function in $L^1(\mu)$ induces a bounded linear functional on n$L^\infty(\mu)$, but the dual space of $L^\infty(\mu)$ is in general larger and difficult to describe. Fortunately, it seldom comes up in applications, see [HS] for further information.

Theorem 4.21. *Let $(X, \mathcal{M}, \mu)$ be σ-finite. Then for $\Lambda \in L^1(\mu)'$, there exists some $g \in L^\infty(\mu)$ such that $\Lambda = \Lambda_g$.*

I let you provide a proof of this theorem.

4.5 Strong Convergence of L^p-sequences

Results concerning the L^1-convergence of sequences of functions are discussed in Chapter 1. Here we present two results on the convergence of L^p-functions.

Proposition 4.22 (Brezis-Lieb Lemma). *Consider $L^p(\mu)$, $1 \leq p < \infty$. Suppose that $\|f_n\|_p \leq M$ and $f_n \to f$ a.e.. Then*

$$\lim_{n\to\infty} \int \left| |f_n|^p - |f_n - f|^p - |f|^p \right| d\mu = 0.$$

Proof. Note the elementary inequality (see below), for $p > 1$,

$$\left| |a+b|^p - |a|^p \right| \leq \varepsilon |a|^p + C_\varepsilon |b|^p , \quad a, b \in \mathbb{R}.$$

Taking $g_n = f_n - f$ as a and f as b,

$$\left| |f+g_n|^p - |g_n|^p \right| \leq \varepsilon |g_n|^p + C_\varepsilon |f|^p ,$$

or,

$$-\varepsilon |g_n|^p - C_\varepsilon |f|^p \leq |f+g_n|^p - |g_n|^p \leq \varepsilon |g_n|^p + C_\varepsilon |f|^p .$$

we have

$$-\varepsilon\,|g_n|^p - (C_\varepsilon + 1)\,|f|^p \le |f + g_n|^p - |g_n|^p - |f|^p \le \varepsilon\,|g_n|^p + (C_\varepsilon - 1)\,|f|^p\,,$$

which implies

$$|\,|f + g_n|^p - |g_n|^p - |f|^p\,| \le \varepsilon\,|g_n|^p + (1 + C_\varepsilon)\,|f|^p\,,$$

and

$$(|\,|f + g_n|^p - |g_n|^p - |f|^p\,| - \varepsilon\,|g_n|^p\,)_+ \le (1 + C_\varepsilon)\,|f|^p\,.$$

By assumption the function $\Phi_n \equiv ||f + g_n|^p - |g_n|^p - |f|^p| \to 0$. By Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int (|\,|f + g_n|^p - |g_n|^p - |f|^p\,| - \varepsilon\,|g_n|^p)_+ \,d\mu = 0.$$

Using

$$\int (F - G) = \int (F - G)_+ - \int (F - G)_- \le \int (F - G)_+,$$

we have

$$\overline{\lim_{n\to\infty}} \int |\,|f + g_n|^p - |g_n|^p - |f|^p\,|\,d\mu \le \varepsilon \,\overline{\lim_{n\to\infty}} \int |g_n|^p \,d\mu.$$

As

$$\begin{aligned}\int |g_n|^p \,d\mu = \int |f_n - f|^p \,d\mu &\le 2^p \left(\int |f_n|^p \,d\mu + \int |f|^p \,d\mu \right) \\ &\le 2^{p+1} M^p,\end{aligned}$$

where Fatou's lemma is used in the last step, we conclude that

$$\overline{\lim_{n\to\infty}} \int |\,|f + g_n|^p - |g_n|^p - |f|^p\,|\,d\mu \le 2^{p+1} M^p \varepsilon,$$

and the desired result follows by letting $\varepsilon \downarrow 0$. □

I leave the case $p = 1$ as exercise. Note that this proposition tells us what is missing in Fatou's lemma.

We provide a proof of the elementary inequality used in the proof above. We have

$$\begin{aligned}|a + b|^p - |a|^p &= \int_0^1 \frac{d}{dt}\,|a + tb|^p \,dt \\ &= p \int_0^1 |a + tb|^{p-2}\,(a + tb) b \,dt\ .\end{aligned}$$

Therefore,

$$
\begin{aligned}
\left||a+b|^p - |a|^p\right| &\le p\int_0^1 (|a|+|b|)^{p-1}\,|b|\;dt \\
&\le p2^{p-1}\left(|a|^{p-1}+|b|^{p-1}\right)|b| \\
&\le p2^{p-1}\,|a|^{p-1}\,|b| + p2^{p-1}\,|b|^p \\
&\le \varepsilon\,|a|^p + C_\varepsilon\,|b|^p\,,
\end{aligned}
$$

where in the last step we have used Young's inequality

$$
xy \le \frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta}, \quad \frac{1}{\alpha}+\frac{1}{\beta} = 1,\ x, y \ge 0
$$

taking $x = |a|^{p-1}, y = |b|$ and suitable α, β

Next we have the Vitali convergence theorem. This theorem is used when a dominator is not available so Lebesgue's dominated convergence theorem cannot be applied.

Theorem 4.23 (Vitali's Convergence Theorem). *Let $\mu(X) < \infty$ and $\{f_n\} \subset L^1(\mu)$ satisfy*

1. *$f_n \to f$ a.e.,*
2. *f_n and f are finite a.e., and*
3. *$\{f_n\}$ is uniformly integrable.*

Then $f_n \to f$ in $L^1(\mu)$.

A sequence (or a set) of integrable functions is called *uniformly integrable* if for every $\varepsilon > 0$, there exists some δ such that

$$
\int_E |f_n|\;d\mu < \varepsilon \qquad \forall n \text{ whenever } E \text{ is measurable with } \mu(E) < \delta.
$$

Proof. By (3), there exists a $\delta \le \varepsilon$ such that the above estimate holds. For this ε, (1), (2) and $\mu(X) < \infty$ enable us to apply Egorov theorem to find some E_1, $\mu(E_1) < \varepsilon$, such that $f_n \to f$ uniformly on $X \setminus E_1$, that is,

$$
\lim_{n\to\infty} \|f_n - f\|_{L^\infty(X\setminus E_1)} = 0.
$$

It follows that

$$
\begin{aligned}
\int |f_n - f|\;d\mu &= \int_{X\setminus E_1} |f_n - f|\;d\mu + \int_{E_1} |f_n - f|\;d\mu \\
&\le \mu(X\setminus E_1)\,\|f_n - f\|_{L^\infty(X\setminus E_1)} + 2\varepsilon.
\end{aligned}
$$

Note that $\int_{E_1} |f|\, d\mu \le \varepsilon$ by Fatou's lemma. So

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int |f_n - f|\, d\mu &\le \mu(X \setminus E_1) \overline{\lim_{n\to\infty}} \|f_n - f\|_{L^\infty(X\setminus E_1)} + 2\varepsilon \\ &= 2\varepsilon,\end{aligned}$$

and

$$\overline{\lim_{n\to\infty}} \int |f_n - f|\, d\mu = 0.$$

□

4.6 Weak Convergence of L^p-sequences

Weak convergence is a concept in a normed space. We call a sequence $\{x_n\}$ *converges weakly* to x in the normed space X, written as $x_n \rightharpoonup x$, if $\Lambda x_n \to \Lambda x$ for all $\Lambda \in X'$. The usual convergence in norm is sometimes called *strong convergence* in contrast with this new notion of convergence. It is clear that $x_n \to x$ implies $x_n \rightharpoonup x$ but the converse is not always true. In fact, for each $\Lambda \in X'$,

$$|\Lambda x_n - \Lambda x| = |\Lambda(x_n - x)| \le \|\Lambda\|\, \|x_n - x\| \to 0$$

as $x_n \to x$ in norm. The weak limit is also unique, for, if $x_n \rightharpoonup x_1$ and $x_n \rightharpoonup x_2$, then $z = x_1 - x_2$ satisfies $\Lambda z = 0\ \forall z \in X$. As a consequence of Hahn-Banach theorem, there always exists a bounded linear functional Λ_1, satisfying $\Lambda_1 z = \|z\|$ (for every given z), $\|z\| = \Lambda_1 z = 0$, and $x_1 = x_2$.

When adapted to the L^p-setting, by duality we know that $f_n \rightharpoonup f$ in $L^p(\mu)$, $1 < p < \infty$, if and only if

$$\int f_n g\, d\mu \to \int f g\, d\mu, \quad \forall g \in L^q(\mu).$$

It is also valid for $p = 1$ when μ is σ-finite.

Example 4.1. Let f_n be a continuous function in $[0,1]$ whose support $[a_n, b_n]$ shrinks to $x_0 \in (0,1)$ as $n \to \infty$. We also require $\|f_n\|_p = 1$. Then $f_n \rightharpoonup 0$ in $L^p(0,1)$. For, as all L^q-functions vanishing near x_0 forms a dense subset of $L^q(0,1)$ and $\int f_n g\, dx = 0$ for such g, we have $f_n \rightharpoonup 0$ in $L^p(0,1)$, $1 < p < \infty$. On the other hand, ais $\|f_n - 0\|_p = \|f_n\|_p = 1$, $\{f_n\}$ does not converge to 0 strongly.

Example 4.2. Let $f_n(x) = \sin 2n\pi x$, $x \in [0,1]$. Then $f_n \rightharpoonup$ in $L^p(0,1)$ for $1 \le p < \infty$. To see this, let $g \in C[0,1]$ first. We need to show that

$$\int_0^1 \sin 2n\pi x g(x)\, dx \to 0 \qquad \text{as } n \to \infty.$$

We have

$$\begin{aligned}\int_0^1 \sin 2n\pi x g(x)\,dx &= \int_0^{2n\pi} \sin y g\left(\frac{y}{2n\pi}\right)\frac{dy}{2n\pi}\\ &= \frac{1}{2n\pi}\sum_{k=1}^{n}\int_{2(k-1)\pi}^{2k\pi} \sin y g\left(\frac{y}{2n\pi}\right)\,dy\\ &= \frac{1}{2n\pi}\sum_{k=1}^{n}\int_0^{2\pi} \sin z g\left(\frac{k-1}{n}+\frac{z}{2n\pi}\right)\,dz.\end{aligned}$$

As $g \in C[0,1]$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ such that $|g(x)-g(y)| < \varepsilon$ if $|x-y| < \delta$. So, for all n large such that $\left|\frac{z}{2n\pi}\right| < \delta$, we have

$$\begin{aligned}&\left|\int_0^{2\pi} \sin z g\left(\frac{k-1}{n}+\frac{z}{2n\pi}\right)\,dz\right|\\ &= \left|\int_0^{2\pi} \sin z\left(g\left(\frac{k-1}{n}+\frac{z}{2n\pi}\right)-g\left(\frac{k-1}{n}\right)\right)\,dz\right| \quad \left(use \int_0^{2\pi}\sin z\,dz = 0\right)\\ &\leq 2\pi\varepsilon\ .\end{aligned}$$

We have

$$\left|\int_0^1 \sin 2n\pi x g(x)\,dx\right| \leq \frac{2\pi\varepsilon\times n}{2n\pi} = \varepsilon,$$

and the desired result holds.

As $C[0,1]$ is dense in $L^q[0,1]$, for $g \in L^q(0,1)$, and $\varepsilon > 0$, we can find a $g_1 \in C[0,1]$ such that $\|g-g_1\|_q < \varepsilon$. Then

$$\begin{aligned}\left|\int_0^1 \sin 2n\pi x g(x)\,dx\right| &\leq \left|\int_0^1 \sin 2n\pi x (g-g_1)(x)\,dx\right| + \left|\int_0^1 \sin 2n\pi x g_1(x)\,dx\right|\\ &\leq \|\sin 2n\pi x\|_p \|g-g_1\|_q + \left|\int_0^1 \sin 2n\pi x g_1(x)\,dx\right|.\end{aligned}$$

As $|\sin 2n\pi x| \leq 1$,

$$\left|\int_0^1 \sin 2n\pi x g(x)\,dx\right| \leq \|g-g_1\|_q \left|\int_0^1 \sin 2n\pi x g(x)\,dx\right|.$$

It follows that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\left|\int_0^1 \sin 2n\pi x g(x)\,dx\right| &\leq \varepsilon + \overline{\lim_{n\to\infty}}\left|\int_0^1 \sin 2n\pi x g_1(x)\,dx\right|\\ &\leq \varepsilon.\end{aligned}$$

We conclude that $\{\sin 2n\pi x\}$ converges weakly to 0 as $n \to \infty$. Clearly the convergence is not strong.

We examine the properties of weakly convergent L^p-sequences.

Proposition 4.24. *Let $\{f_n\}$ be a weakly convergent sequence in some $L^p(\mu)$, $1 \le p \le \infty$. Then $\{f_n\}$ is bounded in L^p-norm.*

This is in fact contained in a general result in functional analysis. Let $\{x_n\}$ be a weakly convergent sequence in the normed space X. It is always true that there is some M such that $\|x_n\| \le M$ for all n. This is called the uniform boundedness principle. In the following we give a proof which applies to $L^p(\mu)$ for $1 < p < \infty$. The cases $p = 1, \infty$ are left as exercises. If you have learnt the uniform boundedness principle, you may skip the following proof, for the proof of the general case is more transparent than this special one.

Proof. Suppose $f_n \rightharpoonup f$ for some f but $\{f_n\}$ is unbounded. By throwing away other f_n's and relabeling the indices, we may assume $\|f_n\|_p \ge 4^n$ for all n. Let $\alpha_n \in (0,1]$ be such that $\widetilde{f}_n = \alpha_n f_n$ satisfies $\|\widetilde{f}_n\|_p = 4^n$. Let $h_n = |\widetilde{f}_n|^{p-1}\widetilde{f}_n / \|\widetilde{f}_n\|^{p-1}$. It is readily checked that $\|h_n\|_q = 1$. We let

$$h = \sum_{j=1}^{\infty} \frac{1}{3^j} \sigma_j h_j,$$

where $\sigma_j \in \{-1, 1\}$ are to be chosen later. We claim that $h \in L^q(\mu)$. For, looking at the difference of the partial sums

$$\begin{aligned}
\left\| \sum_{j=1}^{n} \frac{1}{3^j}\sigma_j h_j - \sum_{j=1}^{m} \frac{1}{3^j}\sigma_j h_j \right\|_q &= \left\| \sum_{m+1}^{n} \frac{1}{3^j}\sigma_j h_j \right\|_q \\
&\le \sum_{j=m+1}^{n} \frac{1}{3^j} \|h_j\|_q \\
&= \frac{1}{2} \times \frac{1}{3^m} \to 0 \quad \text{ as } m \to \infty.
\end{aligned}$$

Thus the partial sums converges in L^q-norm to h. Let Λ_1 be the bounded linear functional on $L^p(\mu)$ induced by h,

$$\Lambda_1 g = \int gh \, d\mu, \quad g \in L^p(\mu).$$

We have

$$\begin{aligned}
\Lambda_1 \widetilde{f}_n &= \int \widetilde{f}_n h \, d\mu \\
&= \sum_{j=1}^{\infty} \frac{1}{3^j} \sigma_j \int h_j \widetilde{f}_n \, d\mu \\
&= \sum_{j=1}^{n} \frac{1}{3_j} \sigma_j \int h_j \widetilde{f}_n \, d\mu + \sum_{n+1}^{\infty} \frac{1}{3^j} \sigma_j \int h_j \widetilde{f}_n \, d\mu.
\end{aligned}$$

The second term in the right hand side is controlled by

$$\begin{aligned}
\sum_{n+1}^{\infty} \frac{1}{3^j} \sigma_j \int h_j \widetilde{f}_n \, d\mu &\le \sum_{n+1}^{\infty} \frac{1}{3^j} \left\| h_j \right\|_q \left\| \widetilde{f}_n \right\| \\
&= \sum_{n+1}^{\infty} \frac{1}{3^j} 4^n = \frac{1}{2} \left(\frac{4}{3} \right)^n .
\end{aligned}$$

Now, we set $\sigma_1 = 1$ and take σ_n (after $\sigma_2, \dots, \sigma_{n-1}$ are fixed) to be 1 or -1 such that $\sum_{j=1}^{n-1} \frac{1}{3^j} \sigma_j \int h_j \widetilde{f}_n \, d\mu$ and $\sigma_n \int h_n \widetilde{f}_n \, d\mu$ have the same sign. Then

$$\sum_{j=1}^{n} \frac{1}{3^j} \sigma_j \int h_j \widetilde{f}_n \, d\mu \ge \frac{1}{3^n} \int h_n \widetilde{f}_n \, d\mu.$$

Combining these two estimates, we get

$$\begin{aligned}
|\Lambda_1 f_n| &\ge \frac{1}{3^n} \int \widetilde{f}_n h_n \, d\mu - \frac{1}{2} \left(\frac{4}{3} \right)^n \\
&= \frac{1}{3^n} \left\| \widetilde{f}_n \right\|_p - \frac{1}{2} \left(\frac{4}{3} \right)^n \\
&= \frac{1}{2} \left(\frac{4}{3} \right)^n \to \infty \qquad \text{as } n \to \infty.
\end{aligned}$$

So

$$\Lambda_1 f_n = \frac{1}{\alpha_n} \Lambda_1 \widetilde{f}_n \to \infty \quad \text{as } n \to \infty, \text{ too,}$$

but this is in conflict with $\Lambda_1 f_n \to \Lambda_1 f$. We conclude that $\{f_n\}$ must be bounded.

□

Next, we show that the L^p-norm is lower semicontinuous with respect to weak convergence.

Proposition 4.25. *Let $\{f_n\}$ be weakly convergent to f in $L^p(\mu)$, $1 \le p < \infty$.*

Then
$$\|f\|_p \leq \underline{\lim}_{n\to\infty} \|f_n\|_p .$$

This proposition looks like Fatou's lemma, but be cautious that the assumption of almost everywhere convergence is now replaced by weak convergence.

Proof. Let $1 < p < \infty$. As $f \in L^p(\mu)$, the function $g = |f|^{p-2} f \in L^q(\mu)$. Let Λ_1 be the bounded linear functional induced by g. We have
$$\begin{aligned}\|f\|_p^p &= \Lambda_1 f \\ &= \lim_{n\to\infty} \Lambda_1 f_n \\ &\leq \|\Lambda_1\| \underline{\lim}_{n\to\infty} \|f_n\|_p .\end{aligned}$$

Noting that by duality,
$$\|\Lambda_1\| = \|g\|_q = \|f\|_p^{\frac{p}{q}} ,$$
so
$$\|f\|_p^p \leq \|f\|_p^{\frac{p}{q}} \underline{\lim}_{n\to\infty} \|f_n\|_p ,$$
and the result follows. When $p = 1$, use the function $g = \mathrm{sgn} f = f/|f|$ and argue similarly. □

This proposition is still valid for $p = \infty$ when μ is σ-finite. Supply a proof for yourself.

Proposition 4.26. *Let $\{f_n\}$ be weakly convergent to f in $L^p(\mu)$, $1 \leq p \leq \infty$. There is $\{g_n\}$ in which each g_n is a convex combinations of $\{f_n\}$ such that $g_n \to f$ in $L^p(\mu)$ strongly.*

By a convex combination we mean $g = \sum_j \lambda_j f_{n_j}$ where $\lambda_j \in (0,1]$, $j = 1, \cdots, N$, for some N satisfying $\sum_{j=1}^N \lambda_j = 1$.

This result, called Mazur's theorem, is a general one. In fact, let $\{x_n\}$ be weakly convergent to x in some normed space X. Then x belongs to the closure of the convex hull of $\{x_n\}$. The convex hull of a set $S \subset X$ is
$$Co(S) = \left\{ y : y = \sum_j^n \lambda_j x_j,\ x_j \in S,\ \lambda_j \in (0,1],\ \sum_j^n \lambda_j = 1 \text{ for some } n \right\}.$$

The convex hull of S is the smallest convex set containing S. It is not hard to show that its closure, $\overline{Co(S)}$, is also convex. Actually it is the smallest closed, convex set containing S.

In the following we give a proof tailored to $L^p(\mu)$ for $1 < p < \infty$. Recall that $L^p(\mu)$ is uniformly convex for p in this range. In a general uniformly convex space X, let K be a closed, convex set in X and x_0 lie outside K. The projection property asserts that there exists $z \in K$ such that

$$\|z - x_0\| \le \|x - x_0\|\,, \quad \forall x \in K.$$

The proof of this fact is the same as the special case where K is a closed subspace, see Proposition 4.18.

Proof. Let $K = \overline{Co(\{f_n\})}$. We want to show that $f \in K$. Assume on the contrary that $f \notin K$. Then by uniform convexity, we can find some $h \in K$ such that

$$\|h - f\|_p \le \|g - f\|_p\,, \quad \forall g \in L^p(\mu).$$

That means the function

$$\varphi(t) = \|(1-t)h + tg - f\|_p^p\,, \quad g \in L^p(\mu),\ t \in [0,1],$$

has a minimum at $t = 0$. So

$$\varphi'(0) = p\int |h-f|^{p-2}\,(h-f)(g-h)\,d\mu \ge 0.$$

Taking $g = f_n \in K$,

$$\int |h-f|^{p-2}\,(h-f)(f_n-h)\,d\mu \ge 0.$$

Observing that $h - f \in L^p(\mu)$ so $|h-f|^{p-2}\,(h-f) \in L^q(\mu)$ and $f_n \rightharpoonup f$ in $L^p(\mu)$, we have

$$\int |h-f|^{p-2}\,(h-f)f_n\,d\mu \to \int |h-f|^{p-2}\,(h-f)f\,d\mu.$$

It follows that

$$\int |h-f|^{p-2}\,(h-f)(f-h)\,d\mu \ge 0,$$

that is,

$$\int |h-f|^p\ d\mu \le 0$$

which forces $f = h \in K$, contradiction holds. □

A nice property that makes weak convergence so important is the following "weak Bolzano-Weierstrass property". Recall that the Bolzano-Weierstrass property asserts that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Theorem 4.27. *Let $\{f_n\}$ be bounded in $L^p(\mu)$, $1 < p < \infty$. There exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightharpoonup f$ for some f in $L^p(\mu)$.*

Proof. We prove this theorem by further assuming that $L^q(\mu)$ is separable. The general case needs more knowledge from functional analysis, see *any* book on functional analysis. See, for instance, chapter 10 in Peter Lax "Functional Analysis".

Let $\{g_k\}$ be a dense subset of $L^q(\mu)$. By taking a Cantor diagonal process, we can extract a subsequence $\{f_{n_k}\}$ from $\{f_n\}$ such that $\lim_{k\to\infty}\int f_{n_k} g_m \, d\mu$ exists for each g_m. For any $g \in L^q(\mu)$, we can find $g_m \to g$ in $L^q(\mu)$. Then

$$\begin{aligned}\left|\int f_{n_k} g_m \, d\mu - \int f_{n_k} g_\ell \, d\mu\right| &= \left|\int f_{n_k}(g_m - g_\ell)\, d\mu\right| \\ &\le \|f_{n_k}\|_p \|g_m - g_\ell\|_q \\ &\le M \|g_m - g_\ell\|_q ,\end{aligned}$$

where $\|f_n\|_p \le M$ by assumption. So $\int f_{n_k} g_m \, d\mu$ is a Cauchy sequence for each f_{n_k} and $\lim_{k\to\infty}\int f_{n_k} g \, d\mu$ exists for each $g \in L^q(\mu)$. We define

$$\Lambda g = \lim_{k\to\infty}\int f_{n_k} g \, d\mu.$$

It is readily checked that the definition is independent of the sequence $\{g_m\}$ approximating g and $|\Lambda g| \le M\|g\|_q$, so $\Lambda \in L^q(\mu)'$. By duality, there exists an $f \in L^p(\mu)$ such that

$$\int g f \, d\mu = \lim_{k\to\infty}\int g f_{n_k}\, d\mu,$$

that is,

$$f_{n_k} \rightharpoonup f \text{ in } L^p(\mu).$$

$\square$

We point out that this theorem is not true when $p = 1$. Consider the bump functions in Example 4.1 ($p = 1$). We have $\|f_n\|_p = 1$, $\forall n$. If $f_{n_k} \rightharpoonup f$ for some function $f \in L^1(\mu)$, f must be 0. But $g \equiv 1 \in L^\infty(0,1)$ and $\int f_{n_k} g \, dx = \int f_{n_k}\, dx = 1$ does not tend to 0.

Theorem 4.28. *For $p \in (1,\infty)$, let $\{f_n\} \subset L^p(\mu)$ satisfy (i) $\|f_n\|_p \to \|f\|_p$ and (ii) $f_n \rightharpoonup f$ as $n \to \infty$. Then $f_n \to f$ in $L^p(\mu)$.*

You should compare this result with the corollary to Brezis-Lieb lemma, which states that under (i) $\|f_n\|_p \to \|f\|_p$ and (ii) $f_n \to f$ a.e., $f_n \to f$ in $L^p(\mu)$ for $1 \le p < \infty$.

The proof of this theorem depends on the uniform convexity of $L^p(\mu)$. Recall that X is uniformly convex if for any pair of unit vectors x and y, for every

$\varepsilon > 0$, there exists some $\theta \in (0,1)$ depending only on ε such that $\|x-y\| \geq \varepsilon \Rightarrow \left\|\frac{1}{2}(x+y)\right\| < 1-\theta$. When applying to $L^p(\mu)$, in order to show $\|f_n - f\|_p \to 0$, it suffices to show $\left\|\frac{1}{2}(f_n+f)\right\|_p \to 0$.

Proof. When $f = 0$ a.e., $\|f_n - 0\|_p \to \|0\| = 0$ is contained in (i). We assume $f \neq 0$.

As $f_n \rightharpoonup f$, $\frac{1}{2}(f_n+f) \rightharpoonup f$, by Proposition 4.24,

$$\|f\|_p \leq \varliminf_{n\to\infty} \left\|\frac{1}{2}(f_n+f)\right\|_p .$$

On the other hand,

$$\left\|\frac{1}{2}(f_n+f)\right\|_p \leq \frac{1}{2}\|f_n\|_p + \frac{1}{2}\|f\|_p .$$

By (ii),

$$\varlimsup_{n\to\infty} \left\|\frac{1}{2}(f_n+f)\right\|_p \leq \varlimsup_{n\to\infty} \left(\frac{1}{2}\|f_n\|_p + \frac{1}{2}\|f\|_p\right) = \|f\|_p .$$

It forces

$$\lim_{n\to\infty} \left\|\frac{1}{2}(f_n+f)\right\|_p = \|f\|_p .$$

Using (ii) and $\|f\|_p \neq 0$, it immediately implies

$$\lim_{n\to\infty} \left\|\frac{1}{2}\left(\|f_n\|_p + \|f\|_p\right)\right\|_p = 1.$$

By uniform convexity,

$$\left\|\|f_n\|_p - \|f\|_p\right\|_p \to 0.$$

By (ii) again, we arrive at $\|f_n - f\|_p \to 0$ as $n \to \infty$. □

Comments on Chapter 4. Basic knowledge on functional analysis is needed to have a good understanding of this chapter. Apart from [R] is our treatment on the dual space of the L^p-space. Here we use the notion of uniform convexity introduced by Clarkson (1936) who showed that L^p-spaces ($p \in (1,\infty)$) are uniformly convex by the inequalities that bear his name. Our proof of Theorem 4.17, which is taken from the web, is a direct extension of the arguments that establish the self-duality of the L^2-space, see also Lieb-Loss Analysis. Another proof can be found in [HS]. Comparing with the different approach in [R], it does not require the measure to be σ-finite. The standard proof found in [R] and many other books is due to von Neumann. It is elegant but, in my opinion, is too tricky.

We will see it in next chapter. It is nice to have more than one proofs for an important theorem.

A new ingredient is weak convergence. It is not covered in [R], but, in view of its importance in applications, I decide to include it here. Many results could be proved in the more general setting of Banach spaces other than the L^p-spaces. Not to get too much involved in functional analysis, I provide self-contained proofs here or there, and sometimes run into tedious arguments, for instance, the proof of Proposition 4.24, which is just a special case of the uniform boundedness principle. Those who have learnt functional analysis can simply ignore them.