THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5520
Differential Equations & Linear Algebra
Suggested Solution for Assignment 5
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Exercise 6.2 Question 1(a)

Find the general solutions to the following systems of differential equations.

$$\begin{cases} x_1' = x_1 + 2x_2 \\ x_2' = 2x_1 + x_2 \end{cases}$$

Soution: Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then the systems of differential equations can be rewritten as $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Solving the characteristic equation

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} &= 0, \\ \lambda^2 - 2\lambda - 3 &= 0, \\ \lambda &= -1, 3. \end{aligned}$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 3$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

respectively. Therefore the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

□

Exercise 6.2 Question 1(c)

Find the general solutions to the following systems of differential equations.

$$\begin{cases} x_1' = x_1 - 5x_2 \\ x_2' = x_1 - x_2 \end{cases}$$

Soution: Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}.$$

Then the systems of differential equations can be rewritten as $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Solving the characteristic equation

$$\begin{vmatrix} \lambda-1 & 5 \\ -1 & \lambda+1 \end{vmatrix} = 0,$$
$$\lambda^2 + 4 = 0,$$
$$\lambda = \pm 2i.$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = 2i$ and $\lambda_2 = -2i$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$$
$$\xi^{(2)} = \begin{pmatrix} 1-2i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} i$$

respectively. Therefore

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 2 \\ 0 \end{pmatrix} \sin 2t = \begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix}$$
$$x^{(2)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t = \begin{pmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{pmatrix}$$

are two linearly independent solutions and the general solution is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= c_1 \begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} 2\cos 2t + \sin 2t \\ \sin 2t \end{pmatrix} \\ &= \begin{pmatrix} (c_1 + 2c_2)\cos 2t + (c_2 - 2c_1)\sin 2t \\ c_1 \cos 2t + c_2 \sin 2t \end{pmatrix}. \end{aligned}$$

□

Exercise 6.2 Question 1(f)

Find the general solutions to the following systems of differential equations.

$$\begin{cases} x_1' = 4x_1 + x_2 + x_3 \\ x_2' = x_1 + 4x_2 + x_3 \\ x_3' = x_1 + x_2 + 4x_3 \end{cases}$$

Soution: Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

Then the systems of differential equations can be rewritten as $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Solving the characteristic equation

$$\begin{vmatrix} \lambda-4 & -1 & -1 \\ -1 & \lambda-4 & -1 \\ -1 & -1 & \lambda-4 \end{vmatrix} = 0,$$
$$(\lambda-3)^2(\lambda-6) = 0,$$
$$\lambda = 3, 3, 6.$$

For the repeated root $\lambda_1 = \lambda_2 = 3$, there are two linearly independent eigenvectors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_3 = 6$, the associated eigenvector is

$$\xi^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

□

Exercise 6.2 Question 2

Solve the following initial value problem.

$$\begin{cases} x_1' = 9x_1 + 5x_2 \\ x_2' = -6x_1 - 2x_2 \\ x_1(0) = 1, x_2(0) = 2 \end{cases}$$

Soution: Define the matrix $\mathbf{A}$ by

$$\mathbf{A} = \begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}.$$

Then the systems of differential equations can be rewritten as $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Solving the characteristic equation

$$\begin{vmatrix} \lambda - 9 & -5 \\ 6 & \lambda + 2 \end{vmatrix} = 0,$$
$$\lambda^2 - 7\lambda + 12 = 0,$$
$$\lambda = 3, 4.$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = 3$ and $\lambda_2 = 4$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 5 \\ -6 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

respectively. Therefore the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 5 \\ -6 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since $x_1(0) = 1, x_2(0) = 0$, we have $c_1 = -1,\ c_2 = 6$. So

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6e^{4t} - 5e^{3t} \\ -6e^{4t} + 6e^{3t} \end{pmatrix}.$$

□

Exercise 6.2 Question 3(b)

Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 5 & -1 \end{pmatrix}$$

Soution: Solving the characteristic equation

$$\begin{aligned} \begin{vmatrix} \lambda - 1 & 1 \\ -5 & \lambda + 1 \end{vmatrix} &= 0, \\ \lambda^2 + 4 &= 0, \\ \lambda &= \pm 2i. \end{aligned}$$

We find that the eigenvalues of the coefficient matrix are $\lambda_1 = 2i$ and $\lambda_2 = -2i$ and the associated eigenvectors are

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 - 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$$

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 + 2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -2 \end{pmatrix} i$$

respectively. Therefore

$$x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin 2t = \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}$$

$$x^{(2)} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos 2t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 2t = \begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}$$

are two linearly independent solutions and the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}.$$

□

Exercise 6.2 Question 3(d)

Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Soution: Solving the characteristic equation

$$\begin{vmatrix} \lambda-4 & 1 & 1 \\ -1 & \lambda-2 & 1 \\ -1 & 1 & \lambda-2 \end{vmatrix} = 0,$$
$$(\lambda-2)(\lambda-3)^2 = 0,$$
$$\lambda = 2, 3, 3.$$

For $\lambda_1 = 2$, the associated eigenvector is

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For the repeated root $\lambda_2 = \lambda_3 = 3$, there are two linearly independent eigenvectors

$$\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \xi^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

□

Exercise 6.3 Question 1(a)

Find the general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}.$$

Soution: Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-1 & -2 \\ 2 & \lambda+3 \end{vmatrix} = 0,$$
$$(\lambda+1)^2 = 0,$$
$$\lambda = -1, -1.$$

$\lambda = -1$ is a double root and the eigenspace associated with $\lambda = -1$ is of dimension 1 and is spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus

$$x^{(1)} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution. Next, we will find a generalized eigenvector of rank 2. Take $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (\mathbf{A} + \mathbf{I})\eta = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (\mathbf{A}+\mathbf{I})^2\eta = \mathbf{0}.$$

Thus, η is a generalized eigenvector of rank 2. Hence

$$x^{(2)} = e^{-t}(\eta + t\eta_1) = e^{-t}\begin{pmatrix} 1+2t \\ -2t \end{pmatrix}$$

is another solution to the system. Therefore the general solution is

$$\mathbf{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1+2t \\ -2t \end{pmatrix}.$$

□

Exercise 6.3 Question 1(c)

Find the general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda-3 & 1 \\ -1 & \lambda-1 \end{vmatrix} &= 0, \\ (\lambda-2)^2 &= 0, \\ \lambda &= 2, 2. \end{aligned}$$

$\lambda = 2$ is a double root and the eigenspace associated with $\lambda = 2$ is of dimension 1 and is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus

$$x^{(1)} = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a solution. Next, we will find a generalized eigenvector of rank 2. Take $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (\mathbf{A}-2\mathbf{I})\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (\mathbf{A}-2\mathbf{I})^2\eta = \mathbf{0}.$$

Thus, η is a generalized eigenvector of rank 2. Hence

$$x^{(2)} = e^{2t}(\eta + t\eta_1) = e^{2t}\begin{pmatrix} 1+t \\ t \end{pmatrix}$$

is another solution to the system. Therefore the general solution is

$$\mathbf{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1+t \\ t \end{pmatrix}.$$

□

Exercise 6.3 Question 1(d)

Find the general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Soution: Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda+3 & 0 & 4 \\ 1 & \lambda+1 & 1 \\ -1 & 0 & \lambda-1 \end{vmatrix} = 0,$$
$$(\lambda+1)^3 = 0,$$
$$\lambda = -1, -1 - 1.$$

Thus $\mathbf{A}$ has an eigenvalue $\lambda = -1$ of multiplicity 3. we find that the associated eigenspace is of dimension 1 and is spanned by $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. We need to find a generalized eigenvector of rank 3. Let $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (\mathbf{A}+\mathbf{I})\eta = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (\mathbf{A}+\mathbf{I})^2\eta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_3 = (\mathbf{A}+\mathbf{I})^3\eta = \mathbf{0}.$$

Therefore, η is a generalized eigenvector of rank 3. Hence

$$x^{(1)} = e^{-t}\eta_2 = e^{-t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x^{(2)} = e^{-t}(\eta_1 + t\eta_2) = e^{-t}\begin{pmatrix} -2 \\ -1+t \\ 1 \end{pmatrix}$$

$$x^{(3)} = e^{-t}(\eta + t\eta_1 + \frac{t^2}{2}\eta_2) = e^{-t}\begin{pmatrix} 1-2t \\ -t+\frac{t^2}{2} \\ t \end{pmatrix}$$

form a fundamental set of solutions to the system.
Therefore the general solution is

$$\mathbf{x} = e^{-t}\left(c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ -1+t \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1-2t \\ -t+\frac{t^2}{2} \\ t \end{pmatrix}\right).$$

□

Exercise 6.3 Question 1(f)

Find the general solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{pmatrix}.$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda-1 & 0 & 0 \\ 2 & \lambda+2 & 3 \\ -2 & -3 & \lambda-4 \end{vmatrix} &= 0, \\ (\lambda-1)^3 &= 0, \\ \lambda &= 1, 1, 1. \end{aligned}$$

Thus $\mathbf{A}$ has an eigenvalue $\lambda = 1$ of multiplicity 3. we find that the associated eigenspace is of dimension 2 and is spanned by $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$. Thus

$$x^{(1)} = e^t \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \quad x^{(2)} = e^t \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$$

are two independent solutions. Next, We need to find a generalized eigenvector of rank 2. Let $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then

$$\eta_1 = (\mathbf{A} - \mathbf{I})\eta = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \neq \mathbf{0},$$

$$\eta_2 = (\mathbf{A} - \mathbf{I})^2\eta = \mathbf{0}.$$

Therefore, η is a generalized eigenvector of rank 2. Hence

$$x^{(3)} = e^t(\eta + t\eta_1) = e^t \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix}$$

is another solution to the system.
Therefore the general solution is

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + c_3 e^t \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix}.$$

$\square$

Exercise 6.4 Question 1(b)

Find $\exp(\mathbf{A}t)$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda-5 & 4 \\ -2 & \lambda+1 \end{vmatrix} &= 0, \\ \lambda^2 - 4\lambda + 3 &= 0, \\ \lambda &= 1, 3. \end{aligned}$$

For $\lambda_1 = 1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 3$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, and hence

$$\begin{aligned} \exp(\mathbf{A}t) = \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= -\begin{pmatrix} e^t & 2e^{3t} \\ e^t & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2e^{3t} - e^t & 2e^t - 2e^{3t} \\ e^{3t} - e^t & 2e^t - e^{3t} \end{pmatrix}. \end{aligned}$$

□

Exercise 6.4 Question 1(d)

Find $\exp(\mathbf{A}t)$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda & -2 \\ 2 & \lambda \end{vmatrix} &= 0, \\ \lambda^2 + 4 &= 0, \\ \lambda &= \pm 2i. \end{aligned}$$

For $\lambda_1 = 2i$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

For $\lambda_2 = -2i$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$, and hence

$$\begin{aligned}
\exp(\mathbf{A}t) = \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} &= \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{2ti} & 0 \\ 0 & e^{-2ti} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{2}\begin{pmatrix} e^{2ti} & ie^{-2ti} \\ ie^{2ti} & e^{-2ti} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{2ti}+e^{-2ti} & -ie^{2ti}+ie^{-2ti} \\ ie^{2ti}-ie^{-2ti} & e^{2ti}+e^{-2ti} \end{pmatrix} \\
&= \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}.
\end{aligned}$$

□

Exercise 6.4 Question 1(e)

Find $\exp(\mathbf{A}t)$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned}
\begin{vmatrix} \lambda & -3 \\ 0 & \lambda \end{vmatrix} &= 0, \\
\lambda^2 &= 0, \\
\lambda &= 0, 0.
\end{aligned}$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = 0$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable. So we need to find a generalized eigenvector of rank 2. Now we take $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and let

$$\eta_1 = \mathbf{A}\eta = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\eta_2 = \mathbf{A}^2\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 2, we may let

$$\mathbf{Q} = [\eta_1 \ \ \eta] = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and hence

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{Q}\exp(\mathbf{J}t)\mathbf{Q}^{-1}\\ &= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\\ &= \frac{1}{3}\begin{pmatrix} 3 & 3t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\\ &= \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

□

Exercise 6.4 Question 1(f)

Find $\exp(\mathbf{A}t)$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda-1 & 1 \\ 8 & 5 & \lambda+3 \end{vmatrix} = 0,$$

$$\lambda^3+\lambda^2-4\lambda-4=0,$$

$$\lambda = -2, -1, 2.$$

For $\lambda_1 = -2$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$.

For $\lambda_2 = -1$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$.

For $\lambda_3 = 2$, the associated eigenvector is $\xi^{(3)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and hence

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1}\\ &= \begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -4 & -3 & 0 \\ 5 & 4 & -1 \\ 7 & 2 & 1 \end{pmatrix}^{-1}\\ &= \begin{pmatrix} -4e^{-2t} & -3e^{-t} & 0 \\ 5e^{-2t} & 4e^{-t} & -e^{2t} \\ 7e^{-2t} & 2e^{-t} & e^{2t} \end{pmatrix}\cdot\frac{1}{12}\begin{pmatrix} 6 & 3 & 3 \\ -12 & -4 & -4 \\ -18 & -13 & -1 \end{pmatrix}\\ &= \frac{1}{12}\begin{pmatrix} -24e^{-2t}+36e^{-t} & -12e^{-2t}+12e^{-t} & -12e^{-2t}+12e^{-t} \\ 30e^{-2t}-48e^{-t}+18e^{2t} & 15e^{-2t}-16e^{-t}+13e^{2t} & 15e^{-2t}-16e^{-t}+e^{2t} \\ 42e^{-2t}-24e^{-t}-18e^{2t} & 21e^{-2t}-8e^{-t}-13e^{2t} & 21e^{-2t}-8e^{-t}-e^{2t} \end{pmatrix}.\end{aligned}$$

□

Exercise 6.4 Question 1(g)

Find $\exp(\mathbf{A}t)$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Soution: $\exp(\mathbf{A}t) = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$

□

Exercise 6.4 Question 2(a)

Solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ for given $\mathbf{A}$ and $\mathbf{x}_0$.

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}; \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda - 2 & -5 \\ 1 & \lambda + 4 \end{vmatrix} &= 0, \\ \lambda^2 + 2\lambda - 3 &= 0, \\ \lambda &= -3, 1. \end{aligned}$$

For $\lambda_1 = -3$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 1$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}$, and hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} \\ &= \begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -e^{-3t} & -5e^{t} \\ e^{-3t} & e^{t} \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 1 & 5 \\ -1 & -1 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} -e^{-3t} + 5e^{t} & -5e^{-3t} + 5e^{t} \\ e^{-3t} - e^{t} & 5e^{-3t} - e^{t} \end{pmatrix}. \end{aligned}$$

Therefore the solution to the initial problem is

$$\begin{aligned} \mathbf{x} &= \exp(\mathbf{A}t)\mathbf{x}_0 \\ &= \frac{1}{4} \begin{pmatrix} -e^{-3t} + 5e^{t} & -5e^{-3t} + 5e^{t} \\ e^{-3t} - e^{t} & 5e^{-3t} - e^{t} \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 6e^{-3t} - 5e^{t} \\ -6e^{-3t} + e^{t} \end{pmatrix}. \end{aligned}$$

□

Exercise 6.4 Question 2(c)

Solve the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ for given $\mathbf{A}$ and $\mathbf{x}_0$.

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}; \quad \mathbf{x}_0 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} &= 0, \\ (\lambda-1)^2(\lambda+1) &= 0, \\ \lambda &= 1, 1, -1. \end{aligned}$$

For $\lambda_1 = \lambda_2 = 1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = -1$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} \\ &= \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}^{-1} \\ &= -\frac{1}{2} \begin{pmatrix} 2e^{-t} & e^t & e^t \\ -e^{-t} & 0 & -e^t \\ e^{-t} & e^{-t} & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} -2e^{-t} & -2e^{-t}+2e^t & -2e^{-t}+2e^t \\ e^{-t}-e^t & e^{-t}-3e^t & e^{-t}-e^t \\ -e^{-t}+e^t & -e^{-t}+e^t & -e^{-t}-e^t \end{pmatrix}. \end{aligned}$$

Therefore the solution to the initial problem is

$$\begin{aligned} \mathbf{x} &= \exp(\mathbf{A}t)\mathbf{x}_0 \\ &= -\frac{1}{2} \begin{pmatrix} -2e^{-t} & -2e^{-t}+2e^t & -2e^{-t}+2e^t \\ e^{-t}-e^t & e^{-t}-3e^t & e^{-t}-e^t \\ -e^{-t}+e^t & -e^{-t}+e^t & -e^{-t}-e^t \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} e^t+2e^{-t} \\ e^t-e^{-t} \\ -2e^t+e^{-t} \end{pmatrix}. \end{aligned}$$

□

Exercise 6.5 Question 1(a)

For the given matrix $\mathbf{A}$, find the Jordan normal form of $\mathbf{A}$ and the matrix exponential $\exp(\mathbf{A}t)$.

$$\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda - 4 & 1 \\ -1 & \lambda - 2 \end{vmatrix} &= 0, \\ (\lambda - 3)^2 &= 0, \\ \lambda &= 3, 3. \end{aligned}$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = 3$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable. So we need to find a generalized eigenvector of rank 2. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (\mathbf{A} - 3\mathbf{I})\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\eta_2 = (\mathbf{A} - 3\mathbf{I})^2\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 2, we may let

$$\mathbf{Q} = [\eta_1 \ \ \eta] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix},$$

and hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \mathbf{Q}\exp(\mathbf{J}t)\mathbf{Q}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \\ &= e^{3t} \begin{pmatrix} 1 + t & -t \\ t & 1 - t \end{pmatrix}. \end{aligned}$$

□

Exercise 6.5 Question 1(c)

For the given matrix $\mathbf{A}$, find the Jordan normal form of $\mathbf{A}$ and the matrix exponential $\exp(\mathbf{A}t)$.

$$\begin{pmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-5 & 1 & -1 \\ -1 & \lambda-3 & 0 \\ 3 & -2 & \lambda-1 \end{vmatrix} = 0,$$
$$(\lambda-3)^3 = 0,$$
$$\lambda = 3, 3, 3.$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = 3$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable. So we need to find a generalized eigenvector of rank 3. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (\mathbf{A} - 3\mathbf{I})\eta = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\eta_2 = (\mathbf{A} - 3\mathbf{I})^2\eta = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$\eta_3 = (\mathbf{A} - 3\mathbf{I})^3\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$\mathbf{Q} = [\eta_2 \quad \eta_1 \quad \eta] = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix},$$

and hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \mathbf{Q}\exp(\mathbf{J}t)\mathbf{Q}^{-1} \\ &= \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{pmatrix} \cdot e^{3t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 2 & -3 & 0 \end{pmatrix}^{-1} \\ &= e^{3t} \begin{pmatrix} 1+2t & -t & t \\ t+t^2 & 1-\frac{t^2}{2} & \frac{t^2}{2} \\ -3t+t^2 & 2t-\frac{t^2}{2} & 1-2t+\frac{t^2}{2} \end{pmatrix}. \end{aligned}$$

$\square$

Exercise 6.5 Question 1(d)

For the given matrix $\mathbf{A}$, find the Jordan normal form of $\mathbf{A}$ and the matrix exponential $\exp(\mathbf{A}t)$.

$$\begin{pmatrix} -2 & 9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda+2 & 9 & 0 \\ -1 & \lambda-4 & 0 \\ -1 & -3 & \lambda-1 \end{vmatrix} &= 0, \\ (\lambda-1)^3 &= 0, \\ \lambda &= 1, 1, 1. \end{aligned}$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = 1$, but the associated eigenspace is of dimension 2, which is spanned by $\xi^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable.

So we need to find a generalized eigenvector of rank 2. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (\mathbf{A} - \mathbf{I})\eta = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$\eta_2 = (\mathbf{A} - \mathbf{I})^2\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$\mathbf{Q} = [\xi^{(1)} \quad \eta_1 \quad \eta] = \begin{pmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence

$$\begin{aligned} \exp(\mathbf{A}t) &= \mathbf{Q}\exp(\mathbf{J}t)\mathbf{Q}^{-1} \\ &= \begin{pmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \\ &= e^t \begin{pmatrix} 1-3t & -9t & 0 \\ t & 1+3t & 0 \\ t & 3t & 1 \end{pmatrix}. \end{aligned}$$

□

Exercise 6.5 Question 1(e)

For the given matrix $\mathbf{A}$, find the Jordan normal form of $\mathbf{A}$ and the matrix exponential $\exp(\mathbf{A}t)$.

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & 7 \\ -1 & -3 & -7 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda+1 & -1 & -1 \\ -1 & \lambda-2 & -7 \\ 1 & 3 & \lambda+7 \end{vmatrix} &= 0, \\ (\lambda+2)^3 &= 0, \\ \lambda &= -2, -2, -2. \end{aligned}$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = -2$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable. So we need to find a generalized eigenvector of rank 3. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (\mathbf{A} + 2\mathbf{I})\eta = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\eta_2 = (\mathbf{A} + 2\mathbf{I})^2\eta = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\eta_3 = (\mathbf{A} - \mathbf{I})^3\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$\mathbf{Q} = [\xi \quad \eta_1 \quad \eta] = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix},$$

and hence

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{Q}\exp(\mathbf{J}t)\mathbf{Q}^{-1}\\ &= \begin{pmatrix} 1 & 1 & 1\\ -2 & 1 & 0\\ 1 & -1 & 0\end{pmatrix}\cdot e^{-2t}\begin{pmatrix} 1 & t & \frac{t^2}{2}\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 1 & 1\\ -2 & 1 & 0\\ 1 & -1 & 0\end{pmatrix}^{-1}\\ &= e^{-2t}\begin{pmatrix} 1+t+\frac{t^2}{2} & t+t^2 & t+\frac{3t^2}{2}\\ t-t^2 & 1+4t-2t^2 & 7t-3t^2\\ -t+\frac{t^2}{2} & -3t+t^2 & 1-5t+\frac{3t^2}{2}\end{pmatrix}.\end{aligned}$$

□

Exercise 6.6 Question 1(a)

Find a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 3 & -2\\ 2 & -2\end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned}\begin{vmatrix} \lambda-3 & 2\\ -2 & \lambda+2\end{vmatrix} &= 0,\\ \lambda^2-\lambda-2 &= 0,\\ \lambda &= -1, 2.\end{aligned}$$

For $\lambda_1 = -1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1\\ 2\end{pmatrix}$.

For $\lambda_2 = 2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2\\ 1\end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0\\ 0 & 2\end{pmatrix}$.
Therefore a fundamental matrix for the system is

$$\begin{aligned}\mathbf{\Psi}(t) = \mathbf{P}\exp(\mathbf{D}t) &= \begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix} e^{-t} & 0\\ 0 & e^{2t}\end{pmatrix}\\ &= \begin{pmatrix} e^{-t} & 2e^{2t}\\ 2e^{-t} & e^{2t}\end{pmatrix}.\end{aligned}$$

□

Exercise 6.6 Question 1(c)

Find a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 2 & -5\\ 1 & -2\end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-2 & 5 \\ -1 & \lambda+2 \end{vmatrix} = 0,$$
$$\lambda^2+1=0,$$
$$\lambda = \pm i.$$

For $\lambda_1 = i$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} i+2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -i$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} i+2 & 2-i \\ 1 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and hence

$$\begin{aligned}
\exp(\mathbf{A}t) &= \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} \\
&= \begin{pmatrix} i+2 & 2-i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{ti} & 0 \\ 0 & e^{-ti} \end{pmatrix}\begin{pmatrix} i+2 & 2-i \\ 1 & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{2i}\begin{pmatrix} (i+2)e^{ti} & (2-i)e^{-ti} \\ e^{ti} & e^{-ti} \end{pmatrix}\begin{pmatrix} 1 & i-2 \\ -1 & i+2 \end{pmatrix} \\
&= \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix},
\end{aligned}$$

which is a fundamental matrix for the system. □

Exercise 6.6 Question 1(f)

Find a fundamental matrix for the system $\mathbf{x}' = \mathbf{Ax}$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 1 & -3 \\ 3 & -5 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{vmatrix} \lambda-1 & 3 \\ -3 & \lambda+5 \end{vmatrix} = 0,$$
$$(\lambda+2)^2 = 0,$$
$$\lambda = -2, -2.$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = -2$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable. So we need to find a generalized eigenvector of rank 2. Now we take $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (\mathbf{A}+2\mathbf{I})\eta = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\eta_2 = (\mathbf{A}+2\mathbf{I})^2\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 2, we may let

$$Q = [\eta_1 \quad \eta] = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix},$$

Therefore a fundamental matrix for the system is

$$\begin{aligned} \mathbf{\Psi}(t) = \mathbf{Q}\exp(\mathbf{J}t) &= \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix} \cdot e^{-2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} 3 & 1+3t \\ 3 & 3t \end{pmatrix}. \end{aligned}$$

□

Exercise 6.6 Question 1(g)

Find a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned} \begin{vmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda-1 & 1 \\ 8 & 5 & \lambda+3 \end{vmatrix} &= 0, \\ (\lambda+1)(\lambda+2)(\lambda-2) &= 0, \\ \lambda &= -1, -2, 2. \end{aligned}$$

For $\lambda_1 = -1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$.

For $\lambda_2 = -2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}$.

For $\lambda_3 = 2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

Let

$$\mathbf{Q} = \begin{pmatrix} -3 & -4 & 0 \\ 4 & 5 & 1 \\ 2 & 7 & -1 \end{pmatrix},$$

then

$$\mathbf{D} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

Therefore a fundamental matrix for the system is

$$\begin{aligned}\mathbf{\Psi}(t) = \mathbf{Q}\exp(\mathbf{D}t) &= \begin{pmatrix} -3 & -4 & 0 \\ 4 & 5 & 1 \\ 2 & 7 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \\ &= \begin{pmatrix} -3e^{-t} & -4e^{-2t} & 0 \\ 4e^{-t} & 5e^{-2t} & e^{2t} \\ 2e^{-t} & 7e^{-2t} & -e^{2t} \end{pmatrix}.\end{aligned}$$

□

Exercise 6.6 Question 1(j)

Find a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is the following matrix.

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{pmatrix}$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned}\begin{vmatrix} \lambda-3 & -1 & -3 \\ -2 & \lambda-2 & -2 \\ 1 & 0 & \lambda-1 \end{vmatrix} &= 0, \\ (\lambda-2)^3 &= 0, \\ \lambda &= 2.\end{aligned}$$

We see that $\mathbf{A}$ has only one eigenvalue $\lambda = 2$, but the associated eigenspace is of dimension 1, which is spanned by $\xi = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Thus $\mathbf{A}$ is not diagonalizable. So we need to find a generalized eigenvector of rank 3. Now we take $\eta = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and let

$$\eta_1 = (\mathbf{A}-2\mathbf{I})\eta = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\eta_2 = (\mathbf{A}-2\mathbf{I})^2\eta = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\eta_3 = (\mathbf{A}-2\mathbf{I})^3\eta = \mathbf{0}.$$

We see that η is a generalized eigenvector of rank 3, we may let

$$Q = [\eta_2 \;\; \eta_1 \;\; \eta] = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix},$$

then

$$\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

Therefore a fundamental matrix for the system is

$$\begin{aligned}\mathbf{\Psi}(t) = \mathbf{Q}\exp(\mathbf{J}t) &= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \cdot e^{2t} \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} 1 & 2+t & \frac{t^2}{2} + 2t + 1 \\ 2 & 2+2t & t^2 + 2t + 1 \\ -1 & -1-t & -\frac{t^2}{2} - t \end{pmatrix}.\end{aligned}$$

□

Exercise 6.6 Question 2(a)

Find the fundamental matrix $\mathbf{\Psi}$ which satisfies $\mathbf{\Psi}(0) = \mathbf{\Psi}_0$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrices $\mathbf{A}$ and $\mathbf{\Psi}_0$.

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}; \quad \mathbf{\Psi}_0 = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}.$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned}\begin{vmatrix} \lambda - 3 & -4 \\ 1 & \lambda + 2 \end{vmatrix} &= 0, \\ (\lambda+1)(\lambda-2) &= 0, \\ \lambda &= -1, 2.\end{aligned}$$

For $\lambda_1 = -1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, and hence

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} \\ &= \begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -e^{-t} & -4e^{2t} \\ e^{-t} & e^{2t} \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -e^{-t} + 4e^{2t} & -4e^{-t} + 4e^{2t} \\ e^{-t} - e^{2t} & 4e^{-t} - e^{2t} \end{pmatrix}.\end{aligned}$$

Therefore the required fundamental matrix with initial condition is

$$\begin{aligned}\mathbf{\Psi}(t) &= \exp(\mathbf{A}t)\mathbf{\Psi}_0 \\ &= \frac{1}{3}\begin{pmatrix} -e^{-t}+4e^{2t} & -4e^{-t}+4e^{2t} \\ e^{-t}-e^{2t} & 4e^{-t}-e^{2t} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -6e^{-t}+12e^{2t} & 4e^{-t}-4e^{2t} \\ 6e^{-t}-3e^{2t} & -4e^{-t}+e^{2t} \end{pmatrix}.\end{aligned}$$

□

Exercise 6.6 Question 2(c)

Find the fundamental matrix $\mathbf{\Psi}$ which satisfies $\mathbf{\Psi}(0) = \mathbf{\Psi}_0$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the given matrices $\mathbf{A}$ and $\mathbf{\Psi}_0$.

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 7 & -4 \\ -2 & 2 & 1 \end{pmatrix}; \quad \mathbf{\Psi}_0 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & -3 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Soution: Solving the characteristic equation, we have

$$\begin{aligned}\begin{vmatrix} \lambda-3 & 0 & 0 \\ 4 & \lambda-7 & 4 \\ 2 & -2 & \lambda-1 \end{vmatrix} &= 0, \\ (\lambda-3)^2(\lambda-5) &= 0, \\ \lambda &= 3, 3, 5.\end{aligned}$$

For $\lambda_1 = \lambda_2 = 3$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda_3 = 5$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, and hence

$$\begin{aligned}\exp(\mathbf{A}t) &= \mathbf{P}\exp(\mathbf{D}t)\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{3t} & e^{3t} & 0 \\ e^{3t} & 0 & 2e^{5t} \\ 0 & -e^{3t} & e^{5t} \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{3t} & 0 & 0 \\ 2e^{3t}-2e^{5t} & -e^{3t}+2e^{5t} & 2e^{3t}-2e^{5t} \\ e^{3t}-e^{5t} & -e^{3t}+e^{5t} & 2e^{3t}-e^{5t} \end{pmatrix}.\end{aligned}$$

Therefore the required fundamental matrix with initial condition is

$$\begin{aligned}
\boldsymbol{\Psi}(t) &= \exp(\mathbf{A}t)\boldsymbol{\Psi}_0 \\
&= \begin{pmatrix} e^{3t} & 0 & 0 \\ 2e^{3t}-2e^{5t} & -e^{3t}+2e^{5t} & 2e^{3t}-2e^{5t} \\ e^{3t}-e^{5t} & -e^{3t}+e^{5t} & 2e^{3t}-e^{5t} \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & -3 & 1 \\ -1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2e^{3t} & 0 & -e^{3t} \\ 2e^{3t}-2e^{5t} & 5e^{3t}-8e^{5t} & -3e^{3t}+4e^{5t} \\ -e^{5t} & 5e^{3t}-4e^{5t} & -2e^{3t}+2e^{5t} \end{pmatrix}.
\end{aligned}$$

□

Exercise 6.7 Question 1(a)

Use the method of variation of parameters to find a particular solution for each of the following non-homogeneous equations.

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -6e^{5t} \\ 6e^{5t} \end{pmatrix}$$

Soution: Define the matrix $\mathbf{A}$ and the vector $\mathbf{g}(t)$ by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \text{ and } \mathbf{g}(t) = \begin{pmatrix} -6e^{5t} \\ 6e^{5t} \end{pmatrix}.$$

Solving the characteristic equation, we have

$$\begin{aligned}
\begin{vmatrix} \lambda-1 & -2 \\ -4 & \lambda-3 \end{vmatrix} &= 0, \\
\lambda^2 - 4\lambda - 5 &= 0, \\
\lambda &= -1, 5.
\end{aligned}$$

For $\lambda_1 = -1$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$.
Therefore a fundamental matrix for the system is

$$\begin{aligned}
\boldsymbol{\Psi}(t) = \mathbf{P}\exp(\mathbf{D}t) &= \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{5t} \end{pmatrix} \\
&= \begin{pmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix}.
\end{aligned}$$

Now

$$
\begin{aligned}
\boldsymbol{\Psi}^{-1}\mathbf{g} &= \begin{pmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix}^{-1} \begin{pmatrix} -6e^{5t} \\ 6e^{5t} \end{pmatrix} \\
&= \frac{1}{-3e^{4t}} \begin{pmatrix} 2e^{5t} & -e^{5t} \\ -e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} -6e^{5t} \\ 6e^{5t} \end{pmatrix} \\
&= \frac{1}{-3e^{4t}} \begin{pmatrix} -18e^{10t} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 6e^{6t} \\ 0 \end{pmatrix}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int \boldsymbol{\Psi}^{-1}\mathbf{g}\, dt &= \int \begin{pmatrix} 6e^{6t} \\ 0 \end{pmatrix} dt \\
&= \begin{pmatrix} e^{6t} + c_1 \\ c_2 \end{pmatrix}.
\end{aligned}
$$

Therefore a particular solution is

$$
\begin{aligned}
\mathbf{x}_p &= \begin{pmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix} \begin{pmatrix} e^{6t} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -e^{5t} \\ e^{5t} \end{pmatrix}.
\end{aligned}
$$

□

Exercise 6.7 Question 1(c)

Use the method of variation of parameters to find a particular solution for each of the following non-homogeneous equations.

$$
\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 9e^{t} \end{pmatrix}
$$

Soution: Define the matrix $\mathbf{A}$ and the vector $\mathbf{g}(t)$ by

$$
\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \text{ and } \mathbf{g}(t) = \begin{pmatrix} 0 \\ 9e^{t} \end{pmatrix}.
$$

Solving the characteristic equation, we have

$$
\begin{aligned}
\begin{vmatrix} \lambda - 2 & 1 \\ -4 & \lambda + 3 \end{vmatrix} &= 0, \\
\lambda^2 + \lambda - 2 &= 0, \\
\lambda &= -2, 1.
\end{aligned}
$$

For $\lambda_1 = -2$, the associated eigenvector is $\xi^{(1)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

For $\lambda_2 = 1$, the associated eigenvector is $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$, then we have $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$.
Therefore a fundamental matrix for the system is

$$\begin{aligned} \mathbf{\Psi}(t) = \mathbf{P}\exp(\mathbf{D}t) &= \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t} & e^{t} \\ 4e^{-2t} & e^{t} \end{pmatrix}. \end{aligned}$$

Now

$$\begin{aligned} \mathbf{\Psi}^{-1}\mathbf{g} &= \begin{pmatrix} e^{-2t} & e^{t} \\ 4e^{-2t} & e^{t} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 9e^{t} \end{pmatrix} \\ &= \frac{1}{-3e^{-t}} \begin{pmatrix} e^{t} & -e^{t} \\ -4e^{-2t} & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ 9e^{t} \end{pmatrix} \\ &= \frac{1}{-3e^{-t}} \begin{pmatrix} -9e^{2t} \\ 9e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} 3e^{3t} \\ -3 \end{pmatrix} \end{aligned}$$

Thus

$$\begin{aligned} \int \mathbf{\Psi}^{-1}\mathbf{g}\,dt &= \int \begin{pmatrix} 3e^{3t} \\ -3 \end{pmatrix} dt \\ &= \begin{pmatrix} e^{3t} + c_1 \\ -3t + c_2 \end{pmatrix}. \end{aligned}$$

Therefore a particular solution is

$$\begin{aligned} \mathbf{x}_p &= \begin{pmatrix} e^{-2t} & e^{t} \\ 4e^{-2t} & e^{t} \end{pmatrix} \begin{pmatrix} e^{3t} \\ -3t \end{pmatrix} \\ &= \begin{pmatrix} (1-3t)e^{t} \\ (4-3t)e^{t} \end{pmatrix}. \end{aligned}$$

$\square$