Napier's Constant

Theorem 1. *Let*

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n \\ b_n &= \sum_{k=0}^{n} \frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \end{aligned}$$

Then

1. $a_n < b_n$ *for any* $n > 1$.
2. a_n *and* b_n *are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof. Observe that by binomial theorem, we have

$$\begin{aligned} a_n &= \left(1+\frac{1}{n}\right)^n \\ &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\cdots+\frac{1}{n^n} \\ &= 1+1+\frac{1}{2!}\cdot\frac{n-1}{n}+\frac{1}{3!}\cdot\frac{(n-1)(n-2)}{n^2}+\cdots+\frac{1}{n!}\cdot\frac{(n-1)\cdots 1}{n^{n-1}} \\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \\ &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\ &= b_n \end{aligned}$$

for any n. Next we show that a_n and b_n are bounded and monotonic.

Boundedness: For any $n > 1$, we have

$$\begin{aligned} 1 < a_n &< b_n \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &\leq 1 + 1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 1 + 2\left(1 - \frac{1}{2^n}\right) \\ &< 3. \end{aligned}$$

Thus a_n and b_n are bounded.

Monotonicity: The monotonicity of b_n is obvious. We prove that a_n is strictly increasing. For any $n \geq 1$, we have

$$\begin{aligned} a_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\ &< 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \cdots \\ &\quad + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right) + \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right) \\ &= a_{n+1}. \end{aligned}$$

Thus a_n are b_n are strictly increasing.

Alternative proof for monotonicity of a_n: Recall that the arithmetic-geometric mean inequality says that for any positive real numbers $x_1, x_2, \ldots, x_k$, not all equal, we have

$$x_1 x_2 \cdots x_k < \left(\frac{x_1 + x_2 + \cdots + x_k}{k}\right)^k.$$

Taking $k = n + 1$, $x_1 = 1$ and $x_i = 1 + \frac{1}{n}$ for $i = 2, 3, \ldots, n + 1$, we have

$$\begin{aligned} 1 \cdot \left(1 + \frac{1}{n}\right)^n &< \left(\frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1}\right)^{n+1} \\ \left(1 + \frac{1}{n}\right)^n &< \left(1 + \frac{1}{n+1}\right)^{n+1}. \end{aligned}$$

We have proved that both a_n and b_n are bounded and monotonic. Therefore a_n are b_n are convergent by monotone convergence theorem.

Next we prove that a_n and b_n have the same limit. Since $a_n < b_n$ for any $n > 1$, we have

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

On the other hand, for a fixed $m \ge 1$, define a sequence c_n (which depends on m) by

$$\begin{aligned} c_n &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \\ &\quad + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right) \end{aligned}$$

Then for any $n > m$, we have $a_n > c_n$ which implies that

$$\begin{aligned} \lim_{n\to\infty} a_n &\ge \lim_{n\to\infty} c_n \\ &= 1 + 1 + \frac{1}{2!}\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \\ &\quad + \frac{1}{m!}\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right) \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \\ &= b_m. \end{aligned}$$

Observe that m is arbitrary and thus

$$\lim_{n\to\infty} a_n \ge \lim_{m\to\infty} b_m = \lim_{n\to\infty} b_n.$$

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

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