MATH1010 University Mathematics
Continuity and differentiability of functions

1. Let $f(x) = \begin{cases} x + a, & \text{if } x < 2 \\ b, & \text{if } x = 2 \\ 3x - 1, & \text{if } x > 2 \end{cases}$

   Find the values of a and b such that $f(x)$ is continuous at $x = 2$.

2. Let $f(x) = \begin{cases} \dfrac{a \sin 2x}{x}, & \text{if } x < 0 \\ b - a, & \text{if } x = 0 \\ a + \dfrac{\ln(1 + 3x)}{x}, & \text{if } x > 0 \end{cases}$

   Find the values of a and b such that $f(x)$ is continuous at $x = 0$.

3. Let $f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ a + b \ln x, & \text{if } x > 1 \end{cases}$

   Find the values of a and b such that $f(x)$ is differentiable at $x = 1$.

4. For each of the following functions, determine whether it is differentiable at $x = 0$. Find $f'(0)$ if it is.

   (a) $f(x) = \begin{cases} 4x + 1, & \text{if } x < 0 \\ x^2 + 4x, & \text{if } x \geq 0 \end{cases}$

   (b) $f(x) = \begin{cases} x^2 + 3x - 1, & \text{if } x < 0 \\ e^{3x} - 2, & \text{if } x \geq 0 \end{cases}$

   (c) $f(x) = xe^{|x|}$

   (d) $f(x) = x^{\frac{1}{3}}$

   (e) $f(x) = x^{\frac{4}{3}}$

   (f) $f(x) = \begin{cases} \dfrac{\sin^2 x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

   (g) $f(x) = \begin{cases} \dfrac{x}{\ln |x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

5. Let $f(x) = |x| \sin x$.

   (a) Find $f'(x)$ for $x \neq 0$.

   (b) Find $f'(0)$.

   (c) Determine whether $f''(0)$ exists.

6. Let $f(x) = |x| \sin^2 x$.

   (a) Find $f'(x)$ for $x \neq 0$.

   (b) Find $f'(0)$.

   (c) Determine whether $f''(0)$ exists.

Solution:

1. Note that

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (x + a) = 2 + a \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (3x - 1) = 5 \\ f(2) &= b \end{aligned}$$

   Since $f(x)$ is continuous, we have $2 + a = 5 = b$ which implies $a = 3$ and $b = 5$.

2. Note that

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \frac{a \sin 2x}{x} = 2a \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(a + \frac{\ln(1 + 3x)}{x} \right) = a + 3 \\ f(0) &= b - a \end{aligned}$$

   Since $f(x)$ is continuous, we have $2a = a + 3 = b - a$ which implies $a = 3$ and $b = 6$.

3. Note that

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x^2 = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (a + b \ln x) = a \end{aligned}$$

Since $f(x)$ is differentiable, $f(x)$ is continuous and we have $a = 1$. Now

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^-} \frac{(1+h)^2-1}{h} = \lim_{h\to 0^-} \frac{2h+h^2}{h} = 2 \\
\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0^+} \frac{b\ln(1+h)}{h} = b
\end{aligned}$$

Since $f'(0)$ exists, we have $b = 2$.

4. (a) Note that

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} (4x+1) = 1 \\
\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (x^2+4x) = 0
\end{aligned}$$

Thus $f(x)$ is not continuous at $x = 0$. Therefore $f'(0)$ does not exist.

(b)

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{(h^2+3h-1)-(-1)}{h} = \lim_{h\to 0^-} (h+3) = 3 \\
\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{(e^{3h}-2)-(-1)}{h} = 3
\end{aligned}$$

Therefore $f(x)$ is differentiable at $x = 0$ and $f'(0) = 3$.

(c) Observe that

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{he^{|h|}-0}{h} = \lim_{h\to 0} e^{|h|} = 1$$

Therefore $f(x)$ is differentiable at $x = 0$ and $f'(0) = 1$.

(d) Observe that

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{1}{3}}-0}{h} = \lim_{h\to 0} h^{-\frac{2}{3}}$$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$.

(e) Observe that

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{\frac{4}{3}}-0}{h}=\lim_{h\to 0}h^{\frac{1}{3}}=0$$

Therefore $f(x)$ is differentiable at $x=0$ and $f'(0)=0$.

(f) Observe that

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{\sin^2 x}{x}=0\neq f(0)$$

Thus $f(x)$ is not continuous at $x=0$. Therefore $f(x)$ is not differentiable at $x=0$.

(g) Observe that

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{\frac{h}{\ln|h|}-0}{h}=\lim_{h\to 0}\frac{1}{\ln|h|}=0$$

Therefore $f(x)$ is differentiable at $x=0$ and $f'(0)=0$.

5. (a) When $x<0$, $f(x)=-x\sin x$ and $f'(x)=-x\cos x-\sin x$
When $x>0$, $f(x)=x\sin x$ and $f'(x)=x\cos x+\sin x$

(b)

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|\sin h-0}{h}=0$$

(c)

$$\lim_{h\to 0^-}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0^-}\frac{-h\cos h-\sin h}{h}=\lim_{h\to 0^-}\left(-\cos h-\frac{\sin h}{h}\right)=-4$$

$$\lim_{h\to 0^+}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0^+}\frac{h\cos h+\sin h}{h}=\lim_{h\to 0^+}\left(\cos h+\frac{\sin h}{h}\right)=4$$

Thus $f'(x)$ is not differentiable at $x=0$. Therefore $f''(0)$ does not exist.

6. (a) When $x<0$, $f(x)=-x\sin^2 x$ and $f'(x)=-\sin^2 x-2x\sin x\cos x$
When $x>0$, $f(x)=x\sin^2 x$ and $f'(x)=\sin^2 x+2x\sin x\cos x$

(b)

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|\sin^2 h-0}{h}=0$$

(c)

$$
\begin{aligned}
\lim_{h\to 0^-}\frac{f'(h)-f'(0)}{h} &= \lim_{h\to 0^-}\frac{-\sin^2 h-2h\sin h\cos h-0}{h}\\
&= \lim_{h\to 0^-}\left(-\frac{\sin^2 h}{h}-2\sin h\cos h\right)\\
&= 0\\
\lim_{h\to 0^+}\frac{f'(h)-f'(0)}{h} &= \lim_{h\to 0^+}\frac{\sin^2 h+2h\sin h\cos h-0}{h}\\
&= \lim_{h\to 0^+}\left(\frac{\sin^2 h}{h}+2\sin h\cos h\right)\\
&= 0
\end{aligned}
$$

Thus $f'(x)$ is differentiable at $x=0$ and $f''(0)=0$.