MATH1010 University Mathematics

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Limits of sequences

Definition (Infinite sequence of real numbers)

An **infinite sequence of real numbers** is defined by a function from the set of positive integers $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ to the set of real numbers $\mathbb{R}$.

Example (Arithmetic sequence)

An **arithmetic sequence** is a sequence a_n such that $a_{n+1} - a_n = d$ is a constant for any n. The constant d is called the **common difference**. The n-th term of the sequence can be calculated by

$$a_n = a_1 + (n-1)d.$$

Sequence	a_1	d	a_n
$1, 3, 5, 7, 9, \dots$	1	2	$a_n = 2n - 1$
$-4, -1, 2, 5, 8, \dots$	7	3	$a_n = 3n - 7$
$19, 12, 5, -2, -9, \dots$	19	-7	$a_n = 26 - 7n$

Example (Geometric sequence)

A **geometric sequence** is a sequence a_n such that $a_{n+1} = ra_n$ for any n where r is a constant. The constant r is called the **common ratio**. The n-th term of the sequence can be calculated by

$$a_n = a_1 r^{n-1}.$$

Sequence	a_1	r	a_n
$1, 2, 4, 8, 16, \ldots$	1	2	$a_n = 2^{n-1}$
$18, 6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$	18	$\frac{1}{3}$	$a_n = \frac{54}{3^n}$
$12, -6, 3, -\frac{3}{2}, \frac{3}{4}, \ldots$	12	$-\frac{1}{2}$	$a_n = \frac{(-1)^{n-1}24}{2^n}$

Example (Fibonacci sequence)

The **Fibonacci sequence** is the sequence F_n which satisfies

$$\begin{cases} F_{n+2} = F_{n+1} + F_n, & \text{for } n \geq 1 \\ F_1 = F_2 = 1 \end{cases}$$

The first few terms of F_n are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

The value of F_n can be calculated by

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Definition (Limit of sequence)

1. Suppose there exists real number L such that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $|a_n - L| < \epsilon$. Then we say that a_n is **convergent**, or a_n **converges to** L, and write

$$\lim_{n\to\infty} a_n = L.$$

   Otherwise we say that a_n is **divergent**.

2. Suppose for any $M > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $a_n > M$. Then we say that a_n **tends to** $+\infty$ as n tends to infinity, and write

$$\lim_{n\to\infty} a_n = +\infty.$$

   We define a_n **tends to** $-\infty$ in a similar way. Note that a_n is divergent if it tends to $\pm\infty$.

Example (Intuitive meaning of limits of infinite sequences)

a_n	First few terms	Limit
$\dfrac{1}{n^2}$	$1, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots$	0
$\dfrac{n}{n+1}$	$\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$	1
$(-1)^{n+1}$	$1, -1, 1, -1, \ldots$	does not exist
$2n$	$2, 4, 6, 8, \ldots$	does not exist/ $+\infty$
$\left(1+\dfrac{1}{n}\right)^n$	$2, \dfrac{9}{4}, \dfrac{64}{27}, \dfrac{625}{256}, \ldots$	$e \approx 2.71828$
$\dfrac{F_{n+1}}{F_n}$	$1, 2, \dfrac{3}{2}, \dfrac{5}{3}, \ldots$	$\dfrac{1+\sqrt{5}}{2} \approx 1.61803$

Definition (Monotonic sequence)

1. We say that a_n is **monotonic increasing** (**decreasing**) if for any $m < n$, we have $a_m \leq a_n$ ($a_m \geq a_n$). We say that a_n is **monotonic** if a_n is either monotonic increasing or monotonic decreasing.
2. We say that a_n is **strictly increasing** (**decreasing**) if for any $m < n$, we have $a_m < a_n$ ($a_m > a_n$).

Definition (Bounded sequence)

We say that a_n is **bounded** if there exists real number M such that $|a_n| < M$ for any $n \in \mathbb{N}$.

Example (Bounded and monotonic sequence)

a_n	Terms	Bounded	Monotonic	Convergent (Limit)
$\frac{1}{n^2}$	$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$	✓	✓	✓ (0)
$1 - \frac{(-1)^n}{n}$	$2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \ldots$	✓	×	✓ (1)
n^2	$1, 4, 9, 16, \ldots$	×	✓	×
$1 - (-1)^n$	$2, 0, 2, 0, \ldots$	✓	×	×
$(-1)^n n$	$-1, 2, -3, 4, \ldots$	×	×	×

Theorem

If a_n is **convergent**, *then a_n is* **bounded**.

Convergent $\Rightarrow$ Bounded

Note that the converse of the above statement is not correct.

Bounded $\not\Rightarrow$ Convergent

The following theorem is very important and we will discuss it in details later.

Theorem (Monotone convergence theorem)

If a_n is **bounded** *and* **monotonic**, *then a_n is* **convergent**.

Bounded *and* **Monotonic $\Rightarrow$ Convergent**

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*

$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b.$$

Answer: **T**

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$ *and* c *is a real number. Then*

$$\lim_{n\to\infty} ca_n = ca.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

$$\lim_{n\to\infty} a_n b_n = ab.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Answer: **F**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b \neq 0$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = 0$, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **F**

Example

For $a_n = \dfrac{1}{n}$ and $b_n = n$, we have $\lim_{n\to\infty} a_n = 0$ but

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} \frac{1}{n} \cdot n = \lim_{n\to\infty} 1 = 1 \neq 0.$$

Exercise (True or False)

If $\lim_{n\to\infty} a_n = 0$ *and* b_n *is* **convergent**, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **T**

Proof.

$$\begin{aligned} \lim_{n\to\infty} a_n b_n &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n \\ &= 0 \end{aligned}$$

□

Exercise (True or False)

If $\lim_{n\to\infty} a_n = 0$ *and* b_n *is* **bounded**, *then*

$$\lim_{n\to\infty} a_n b_n = 0.$$

Answer: **T**

Caution! The previous proof does not work.

Exercise (True or False)

If a_n^2 is convergent, then a_n is convergent.

Answer: **F**

Example

For $a_n = (-1)^n$, a_n^2 converges to 1 but a_n is divergent.

Exercise (True or False)

If a_n is convergent, then $|a_n|$ is convergent.

Answer: **T**

Exercise (True or False)

If $|a_n|$ *is convergent, then* a_n *is convergent.*

Answer: **F**

Exercise (True or False)

If a_n and b_n are divergent, then $a_n + b_n$ is divergent.

Answer: **F**

Example

The sequences $a_n = n$ and $b_n = -n$ are divergent but $a_n + b_n = 0$ converges to 0.

Exercise (True or False)

If $\lim_{n\to\infty} b_n = +\infty$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **F**

Example

For $a_n = n^2$ and $b_n = n$, we have $\lim_{n\to\infty} b_n = +\infty$ but $\frac{a_n}{b_n} = \frac{n^2}{n} = n$ is divergent.

Exercise (True or False)

If a_n *is convergent and* $\lim_{n\to\infty} b_n = \pm\infty$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **T**

Exercise (True or False)

If a_n *is* **bounded** *and* $\lim_{n\to\infty} b_n = \pm\infty$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Answer: **T**

Exercise (True or False)

Suppose a_n *is bounded. Suppose* b_n *is a sequence and there exists* N *such that* $b_n = a_n$ *for any* $n > N$*. Then* b_n *is bounded.*

Answer: **T**

Exercise (True or False)

Suppose $\lim_{n\to\infty} a_n = a$. *Suppose* b_n *is a sequence and there exists* N *such that* $b_n = a_n$ *for any* $n > N$. *Then*

$$\lim_{n\to\infty} b_n = a.$$

Answer: **T**

Exercise (True or False)

Suppose a_n *and* b_n *are convergent sequences such that* $a_n < b_n$ *for any* n*. Then*

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n.$$

Answer: **F**

Example

The sequences $a_n = 0$ and $b_n = \dfrac{1}{n}$ satisfy $a_n < b_n$ for any n. However

$$\lim_{n\to\infty} a_n \not< \lim_{n\to\infty} b_n$$

because both of them are 0.

Exercise (True or False)

Suppose a_n *and* b_n *are convergent sequences such that* $a_n \leq b_n$ *for any* n. *Then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_n = a$, *then*

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = a.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = a$, *then*

$$\lim_{n\to\infty} a_n = a.$$

Answer: **T**

Exercise (True or False)

If a_n is convergent, then

$$\lim_{n\to\infty}(a_{n+1} - a_n) = 0.$$

Answer: **T**

Exercise (True or False)

If $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, *then* a_n *is convergent.*

Answer: **F**

Example

Let $a_n = \sqrt{n}$. Then $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ and a_n is divergent.

Exercise (True or False)

If $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ *and* a_n *is bounded, then* a_n *is convergent.*

Answer: **F**

Example

$$0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{6}, \frac{2}{6}, \ldots$$

Example

Let $a > 0$ be a positive real number.

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } 0 < a < 1 \end{cases}.$$

Example

$$\begin{aligned}\lim_{n\to\infty}\frac{2n-5}{3n+1} &= \lim_{n\to\infty}\frac{2-\frac{5}{n}}{3+\frac{1}{n}}\\ &= \frac{2-0}{3+0}\\ &= \frac{2}{3}\end{aligned}$$

Example

$$\lim_{n\to\infty} \frac{n^3 - 2n + 7}{4n^3 + 5n^2 - 3} = \lim_{n\to\infty} \frac{1 - \frac{2}{n^2} + \frac{7}{n^3}}{4 + \frac{5}{n} - \frac{3}{n^3}} = \frac{1}{4}$$

Example

$$\begin{aligned}\lim_{n\to\infty}\frac{3n-\sqrt{4n^2+1}}{3n+\sqrt{9n^2+1}} &= \lim_{n\to\infty}\frac{3-\frac{\sqrt{4n^2+1}}{n}}{3+\frac{\sqrt{9n^2+1}}{n}}\\ &= \lim_{n\to\infty}\frac{3-\sqrt{4+\frac{1}{n^2}}}{3+\sqrt{9+\frac{1}{n^2}}}\\ &= \frac{1}{6}\end{aligned}$$

Example

$$
\begin{aligned}
& \lim_{n\to\infty}(n-\sqrt{n^2-4n+1}) \\
= & \lim_{n\to\infty}\frac{(n-\sqrt{n^2-4n+1})(n+\sqrt{n^2-4n+1})}{n+\sqrt{n^2-4n+1}} \\
= & \lim_{n\to\infty}\frac{n^2-(n^2-4n+1)}{n+\sqrt{n^2-4n+1}} \\
= & \lim_{n\to\infty}\frac{4n-1}{n+\sqrt{n^2-4n+1}} \\
= & \lim_{n\to\infty}\frac{4-\frac{1}{n}}{1+\sqrt{1-\frac{4}{n}+\frac{1}{n^2}}} \\
= & 2
\end{aligned}
$$

Example

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\ln(n^4+1)}{\ln(n^3+1)} &= \lim_{n\to\infty} \frac{\ln(n^4(1+\frac{1}{n^4}))}{\ln(n^3(1+\frac{1}{n^3}))} \\
&= \lim_{n\to\infty} \frac{\ln n^4 + \ln(1+\frac{1}{n^4})}{\ln n^3 + \ln(1+\frac{1}{n^3})} \\
&= \lim_{n\to\infty} \frac{4\ln n + \ln(1+\frac{1}{n^4})}{3\ln n + \ln(1+\frac{1}{n^3})} \\
&= \lim_{n\to\infty} \frac{4 + \frac{\ln(1+\frac{1}{n^4})}{\ln n}}{3 + \frac{\ln(1+\frac{1}{n^3})}{\ln n}} \\
&= \frac{4}{3}
\end{aligned}
$$

Squeeze theorem

Theorem (Squeeze theorem)

Suppose a_n, b_n, c_n *are sequences such that* $a_n \leq b_n \leq c_n$ *for any* n *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. *Then* b_n *is convergent and*

$$\lim_{n\to\infty} b_n = L.$$

Theorem

If a_n *is bounded and* $\lim_{n\to\infty} b_n = 0$*, then* $\lim_{n\to\infty} a_n b_n = 0$.

Proof.

Since a_n is bounded, there exists M such that $-M < a_n < M$ for any n. Thus

$$-M|b_n| < a_n b_n < M|b_n|$$

for any n. Now

$$\lim_{n\to\infty} (-M|b_n|) = \lim_{n\to\infty} M|b_n| = 0.$$

Therefore by squeeze theorem, we have

$$\lim_{n\to\infty} a_n b_n = 0.$$

□

Example

Find $\lim_{n\to\infty} \frac{\sqrt{n}+(-1)^n}{\sqrt{n}-(-1)^n}$.

Solution

Since $(-1)^n$ *is bounded and* $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, *we have* $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$ *and therefore*

$$\begin{aligned}\lim_{n\to\infty} \frac{\sqrt{n}+(-1)^n}{\sqrt{n}-(-1)^n} &= \lim_{n\to\infty} \frac{1+\frac{(-1)^n}{\sqrt{n}}}{1-\frac{(-1)^n}{\sqrt{n}}} \\ &= 1\end{aligned}$$

Example

Show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Proof.

Observe that for any $n \geq 3$,

$$0 < \frac{2^n}{n!} = 2\left(\frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdots \frac{2}{n-1}\right)\frac{2}{n} \leq 2 \cdot \frac{2}{n} = \frac{4}{n}$$

and $\lim_{n\to\infty} \frac{4}{n} = 0$. By squeeze theorem, we have

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

□

Monotone convergence theorem

Theorem (Monotone convergence theorem)

If a_n *is* **bounded** *and* **monotonic**, *then* a_n *is* **convergent**.

Bounded *and* **Monotonic** $\Rightarrow$ **Convergent**

Example

Let a_n be the sequence defined by the recursive relation

$$\begin{cases} a_{n+1} = \sqrt{a_n + 1} \text{ for } n \geq 1 \\ a_1 = 1 \end{cases}.$$

Find $\lim_{n\to\infty} a_n$.

n	a_n
1	1
2	1.414213562
3	1.553773974
4	1.598053182
5	1.611847754
10	1.618016542
15	1.618033940

Solution

Suppose $\lim_{n\to\infty} a_n = a$. *Then* $\lim_{n\to\infty} a_{n+1} = a$ *and thus*

$$\begin{aligned} a &= \sqrt{a+1} \\ a^2 &= a+1 \\ a^2 - a - 1 &= 0 \end{aligned}$$

By solving the quadratic equation, we have

$$a = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}.$$

It is obvious that $a > 0$. *Therefore*

$$a = \frac{1+\sqrt{5}}{2} \approx 1.6180339887$$

Solution

The above solution is not complete. The solution is valid only after we have proved that $\lim_{n\to\infty} a_n$ *exists and is positive. This can be done by using* **monotone convergent theorem**. *We are going to show that* a_n *is* **bounded** *and* **monotonic**.

Boundedness

We prove that $1 \leq a_n < 2$ *for all* $n \geq 1$ *by induction.*
(Base case) When $n = 1$, *we have* $a_1 = 1$ *and* $1 \leq a_1 < 2$.
(Induction step) Assume that $1 \leq a_k < 2$. *Then*

$$\begin{aligned} a_{k+1} &= \sqrt{a_k + 1} \geq \sqrt{1+1} > 1 \\ a_{k+1} &= \sqrt{a_k + 1} < \sqrt{2+1} < 2 \end{aligned}$$

Thus $1 \leq a_n < 2$ *for any* $n \geq 1$ *which implies that* a_n *is bounded.*

Solution

Monotonicity

We prove that $a_{n+1} > a_n$ *for any* $n \geq 1$ *by induction.*
(Base case) When $n = 1$, $a_1 = 1$, $a_2 = \sqrt{2}$ *and thus* $a_2 > a_1$.
(Induction step) Assume that

$$a_{k+1} > a_k \text{ (Induction hypothesis).}$$

Then

$$\begin{aligned} a_{k+2} &= \sqrt{a_{k+1}+1} > \sqrt{a_k+1} \text{ (by induction hypothesis)} \\ &= a_{k+1} \end{aligned}$$

This completes the induction step and thus a_n *is strictly increasing.*
We have proved that a_n *is bounded and strictly increasing. Therefore* a_n *is convergent by monotone convergence theorem. Since* $a_n \geq 1$ *for any* n, *we have* $\lim_{n\to\infty} a_n \geq 1$ *is positive.*

Example

Let $a_n = \dfrac{F_{n+1}}{F_n}$ where F_n is the Fibonacci's sequence defined by

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_1 = F_2 = 1 \end{cases}.$$

Find $\lim_{n\to\infty} a_n$.

n	a_n
1	1
2	2
3	1.5
4	1.666666666
5	1.6
10	1.618181818
15	1.618032787
20	1.618033999

Theorem

For any $n \geq 1$,

1. $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$
2. $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$

Proof

1. *When* $n = 1$, *we have* $F_3F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2$. *Assume*

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}.$$

Then

$$\begin{aligned} F_{k+3}F_{k+1} - F_{k+2}^2 &= (F_{k+2} + F_{k+1})F_{k+1} - F_{k+2}^2 \\ &= F_{k+2}(F_{k+1} - F_{k+2}) + F_{k+1}^2 \\ &= -F_{k+2}F_k + F_{k+1}^2 \\ &= (-1)^{k+2} \text{ (by induction hypothesis)} \end{aligned}$$

Therefore $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$ *for any* $n \geq 1$.

Proof.

The proof for the second statement is basically the same. When $n = 1$, we have $F_4F_1 - F_3F_2 = 3 \cdot 1 - 2 \cdot 1 = 1 = (-1)^2$. Assume

$$F_{k+3}F_k - F_{k+2}F_{k+1} = (-1)^{k+1}.$$

Then

$$\begin{aligned}
F_{k+4}F_{k+1} - F_{k+3}F_{k+2} &= (F_{k+3} + F_{k+2})F_{k+1} - F_{k+3}F_{k+2} \\
&= F_{k+3}(F_{k+1} - F_{k+2}) + F_{k+2}F_{k+1} \\
&= -F_{k+3}F_k + F_{k+2}F_{k+1} \\
&= -(-1)^{k+1} \text{ (by induction hypothesis)} \\
&= (-1)^{k+2}
\end{aligned}$$

Therefore $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$ for any $n \geq 1$. □

Theorem

Let $a_n = \dfrac{F_{n+1}}{F_n}$.

1. *The sequence* $a_1, a_3, a_5, a_7, \cdots$, *is strictly increasing.*
2. *The sequence* $a_2, a_4, a_6, a_8, \cdots$, *is strictly decreasing.*

Proof.

For any $k \geq 1$, we have

$$\begin{aligned} a_{2k+1} - a_{2k-1} &= \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{2k}}{F_{2k-1}} = \frac{F_{2k+2}F_{2k-1} - F_{2k+1}F_{2k}}{F_{2k+1}F_{2k-1}} \\ &= \frac{(-1)^{2k}}{F_{2k+1}F_{2k-1}} = \frac{1}{F_{2k+1}F_{2k-1}} > 0 \end{aligned}$$

Therefore $a_1, a_3, a_5, a_7, \cdots$, is strictly increasing. The second statement can be proved in a similar way. □

Theorem

$$\lim_{k\to\infty}(a_{2k+1}-a_{2k})=0$$

Proof.

For any $k \geq 1$,

$$\begin{aligned} a_{2k+1}-a_{2k} &= \frac{F_{2k+2}}{F_{2k+1}}-\frac{F_{2k+1}}{F_{2k}} \\ &= \frac{F_{2k+2}F_{2k}-F_{2k+1}^2}{F_{2k+1}F_{2k}}=\frac{1}{F_{2k+1}F_{2k}} \end{aligned}$$

Therefore

$$\lim_{k\to\infty}(a_{2k+1}-a_{2k})=\lim_{k\to\infty}\frac{1}{F_{2k+1}F_{2k}}=0.$$

□

Theorem

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}$$

Proof

First we prove that $a_n=\frac{F_{n+1}}{F_n}$ *is convergent.*
a_n *is bounded.* *(*$1\leq a_n\leq 2$ *for any* n*.)*
a_{2k+1} *and* a_{2k} *are convergent. (They are bounded and monotonic.)*

$$\lim_{k\to\infty}(a_{2k+1}-a_{2k})=0\Rightarrow\lim_{k\to\infty}a_{2k+1}=\lim_{k\to\infty}a_{2k}$$

It follows that a_n *is convergent and*

$$\lim_{n\to\infty}a_n=\lim_{k\to\infty}a_{2k+1}=\lim_{k\to\infty}a_{2k}.$$

Proof.

To evaluate the limit, suppose $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = L$. Then

$$L = \lim_{n\to\infty} \frac{F_{n+2}}{F_{n+1}} = \lim_{n\to\infty} \frac{F_{n+1} + F_n}{F_{n+1}} = \lim_{n\to\infty} \left(1 + \frac{F_n}{F_{n+1}}\right) = 1 + \frac{1}{L}$$

$$L^2 - L - 1 = 0$$

By solving the quadratic equation, we have

$$L = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}.$$

We must have $L \geq 1$ since $a_n \geq 1$ for any n. Therefore

$$L = \frac{1+\sqrt{5}}{2}.$$

□

Remarks

The limit can be calculate directly using the formula

$$
\begin{aligned}
F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)
\end{aligned}
$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

Theorem

Let

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$b_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Then

1. $a_n < b_n$ *for any* $n > 1$.
2. a_n *and* b_n *are convergent and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$b_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

n	a_n	b_n
1	2	2
5	2.48832	2.716666666666
10	2.593742	2.718281801146
100	2.704813	2.718281828459
100000	2.718268	2.718281828459

The limit of the two sequences is the important Euler's number

$$e \approx 2.71828\,18284\,59045\,23536\ldots.$$

which is also known as the Napier's constant.

Definition (Convergence of infinite series)

We say that an **infinite series**

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is **convergent** if the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$ is convergent. If the infinite series is convergent, then we define

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Limits of functions

Definition (Function)

A real valued **function** on a subset $D \subset \mathbb{R}$ is a real value $f(x)$ assigned to each of the values $x \in D$. The set D is called the **domain** of the function.

Given an expression $f(x)$ in x, the domain D is understood to be taken as the set of all real numbers x such that $f(x)$ is defined. This is called the maximum domain of definition of $f(x)$.

Definition (Graph of function)

Let $f(x)$ is a real valued function. The graph of $f(x)$ is the set

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

Definition

Let $f(x)$ be a real valued function and D be its domain. We say that $f(x)$ is

1. **injective** if for any $x_1, x_2 \in D$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.
2. **surjective** if for any real number $y \in \mathbb{R}$, there exists $x \in D$ such that $f(x) = y$.
3. **bijective** if $f(x)$ is both injective and surjective.

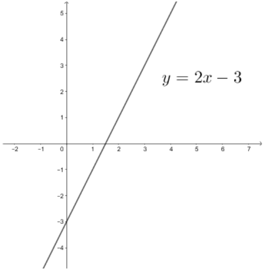
Definition

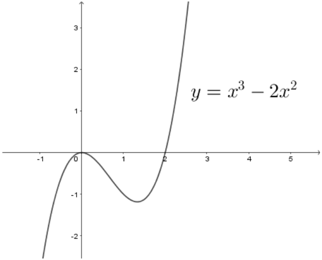
Let $f(x)$ be a real valued function. We say that $f(x)$ is

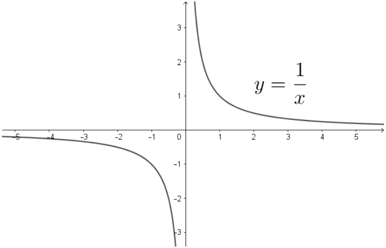
1. **even** if $f(-x) = f(x)$ for any x.
2. **odd** if $f(-x) = -f(x)$ for any x.

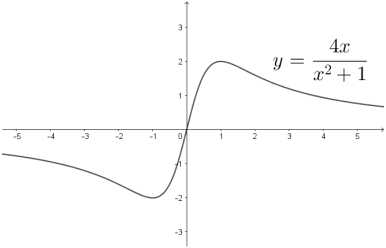
Example

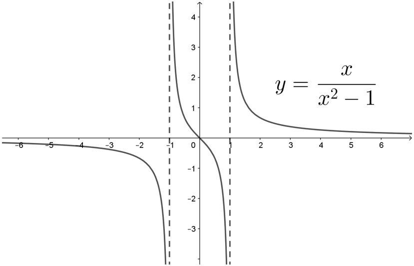
$f(x)$	Domain	Injective	Surjective	Bijective	Even	Odd
$2x-3$	$\mathbb{R}$	✓	✓	✓	×	×
x^3-2x^2	$\mathbb{R}$	×	✓	×	×	×
$\frac{1}{x}$	$x \neq 0$	✓	×	×	×	✓
$\frac{4x}{x^2+1}$	$\mathbb{R}$	×	×	×	×	✓
$\frac{x}{x^2-1}$	$x \neq \pm 1$	×	✓	×	×	✓
$x^2-\frac{1}{x^2}$	$x \neq 0$	×	✓	×	✓	×
$\sqrt{4-x^2}$	$-2 \leq x \leq 2$	×	×	×	✓	×
$\frac{1}{\sqrt{x+4}}$	$x > -4$	✓	×	×	×	×

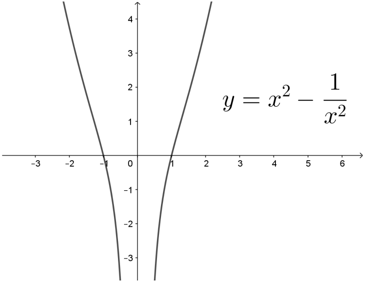

$y = 2x - 3$

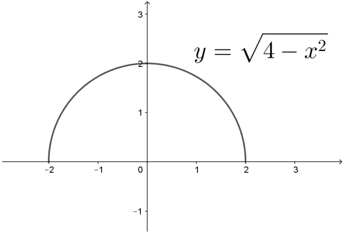

$y = x^3 - 2x^2$

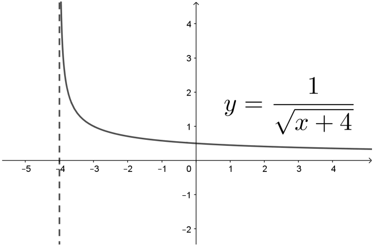

$y = \frac{1}{x}$


$y = \frac{4x}{x^2+1}$


$y = \frac{x}{x^2 - 1}$


$y = x^2 - \frac{1}{x^2}$


$y = \sqrt{4 - x^2}$


$y = \frac{1}{\sqrt{x + 4}}$

Definition (Limit of function)

Let $f(x)$ be a real valued function.

1. We say that a real number l is a limit of $f(x)$ at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

   $$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - l| < \epsilon$$

   and write

   $$\lim_{x \to a} f(x) = l.$$

2. We say that a real number l is a limit of $f(x)$ at $+\infty$ if for any $\epsilon > 0$, there exists $R > 0$ such that

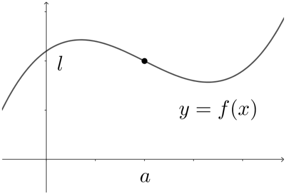
   $$\text{if } x > R, \text{ then } |f(x) - l| < \epsilon$$

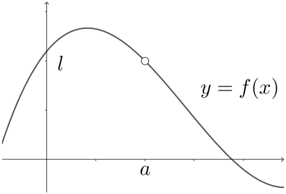
   and write

   $$\lim_{x \to +\infty} f(x) = l.$$

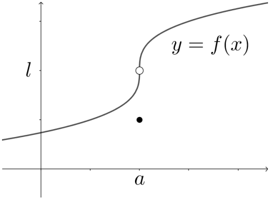
   The limit of $f(x)$ at $-\infty$ is defined similarly.

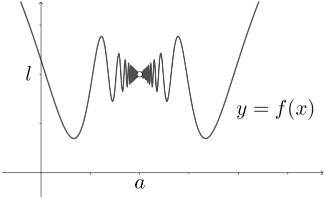
1. Note that for the limit of $f(x)$ at $x = a$ to exist, $f(x)$ may not be defined at $x = a$ and even if $f(a)$ is defined, the value of $f(a)$ does not affect the value of $\lim_{x \to a} f(x)$.
2. The limit of $f(x)$ at $x = a$ may not exist. However the limit is unique if it exists.

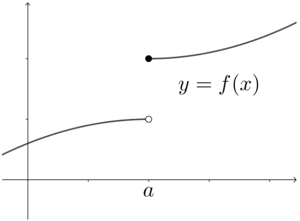


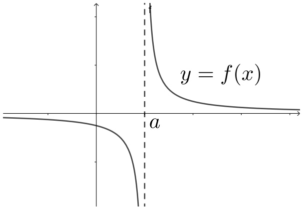


$$\lim_{x \to a} f(x) = l$$



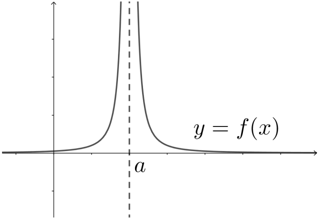


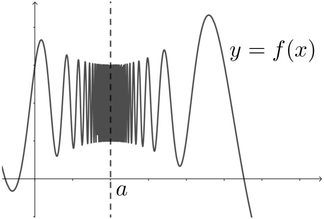




$$\lim_{x \to a} f(x)$$

does not exist





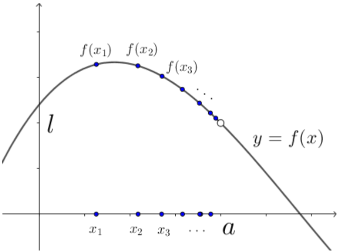
Theorem (Sequential criterion for limits of functions)

Let $f(x)$ be a real valued function. Then

$$\lim_{x\to a} f(x) = l$$

if and only if for any sequence x_n of real numbers with $\lim_{n\to\infty} x_n = a$, we have

$$\lim_{n\to\infty} f(x_n) = l.$$



Theorem

Let $f(x)$, $g(x)$ *be functions such that* $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ *exist and* c *be a real number. Then*

1. $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
3. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$.

Theorem

Let $g(u)$ be a function of u and $u = f(x)$ be a function of x. Suppose

1. $\lim_{x \to a} f(x) = b \in [-\infty, +\infty]$
2. $\lim_{u \to b} g(u) = l$
3. $f(x) \neq b$ *when* $x \neq a$ *or* $g(b) = l$.

Then

$$\lim_{x \to a} (g \circ f)(x) = l.$$

$$x \xrightarrow{f} u = f(x) \xrightarrow{g} (g \circ f)(x) = g(u) = g(f(x))$$

Example

1. $$\lim_{x\to+\infty}\frac{6x^3+2x^2-5}{2x^3-3x+1} = \lim_{x\to+\infty}\frac{6+\frac{2}{x}-\frac{5}{x^3}}{2-\frac{3}{x^2}+\frac{1}{x^3}} = \lim_{y\to 0}\frac{6+2y-5y^3}{2-3y+y^3} = 3$$

2. $$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = e$$

Theorem (Squeeze theorem)

Let $f(x), g(x), h(x)$ be real valued functions. Suppose

1. $f(x) \leq g(x) \leq h(x)$ *for any* $x \neq a$ *on a neighborhood of* a, *and*
2. $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$.

Then the limit of $g(x)$ *at* $x = a$ *exists and* $\lim_{x \to a} g(x) = l$.

Theorem

Suppose

1. $f(x)$ *is bounded, and*
2. $\lim_{x \to a} g(x) = 0$

Then $\lim_{x \to a} f(x)g(x) = 0$.

Exponential, logarithmic and trigonometric functions

Definition (Exponential function)

The **exponential function** is defined for real number $x \in \mathbb{R}$ by

$$\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

1. It can be proved that the two limits in the definition exist and converge to the same value for any real number x.
2. e^x is just a notation for the exponential function. One should not interpret it as 'e to the power x'.

Theorem

For any $x, y \in \mathbb{R}$, *we have*

$$e^{x+y} = e^x e^y.$$

Caution! One cannot use law of indices to prove the above identity. It is because e^x is just a notation for the exponential function and it does not mean 'e to the power x'. In fact we have not defined what a^x means when x is a real number which is not rational.

Theorem

1. $e^x > 0$ *for any real number* x.
2. e^x *is strictly increasing.*

Proof.

1. For any $x > 0$, we have $e^x > 1 + x > 1$. If $x < 0$, then

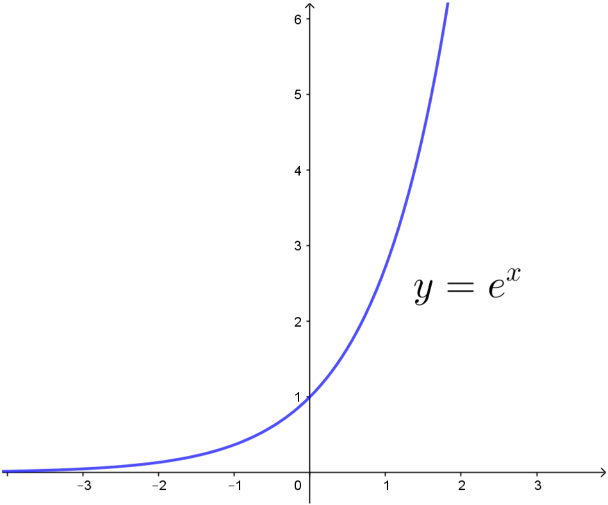
$$\begin{aligned} e^x e^{-x} &= e^{x+(-x)} = e^0 = 1 \\ e^x &= \frac{1}{e^{-x}} > 0 \end{aligned}$$

   since $e^{-x} > 1$. Therefore $e^x > 0$ for any $x \in \mathbb{R}$.

2. Let x, y be real numbers with $x < y$. Then $y - x > 0$ which implies $e^{y-x} > 1$. Therefore

$$e^y = e^{x+(y-x)} = e^x e^{y-x} > e^x.$$

□


$y = e^x$
6
5
4
3
2
1
−3
−2
−1
0
1
2
3

Definition (Logarithmic function)

The **logarithmic function** is the function $\ln : \mathbb{R}^+ \to \mathbb{R}$ defined for $x > 0$ by

$$y = \ln x \text{ if } e^y = x.$$

In other words, $\ln x$ is the inverse function of e^x.

It can be proved that for any $x > 0$, there exists unique real number y such that $e^y = x$.

Theorem

1. $\ln xy = \ln x + \ln y$
2. $\ln \dfrac{x}{y} = \ln x - \ln y$
3. $\ln x^n = n \ln x$ *for any integer* $n \in \mathbb{Z}$.

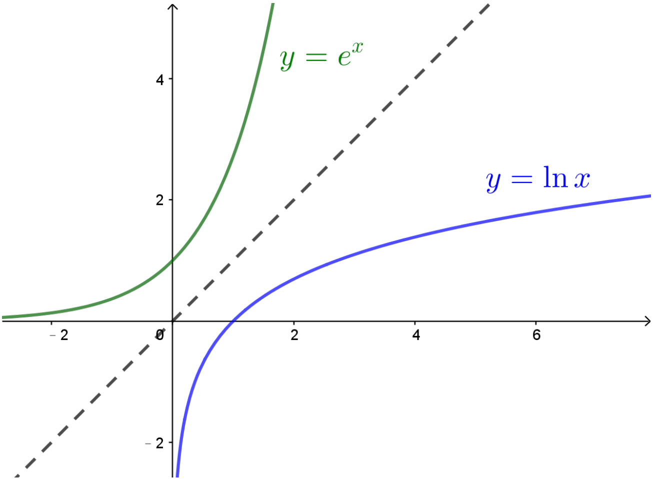
Proof.

1. Let $u = \ln x$ and $v = \ln y$. Then $x = e^u$, $y = e^v$ and we have

$$xy = e^u e^v = e^{u+v} = e^{\ln x + \ln y}$$

which means $\ln xy = \ln x + \ln y$.

Other parts can be proved similarly. □

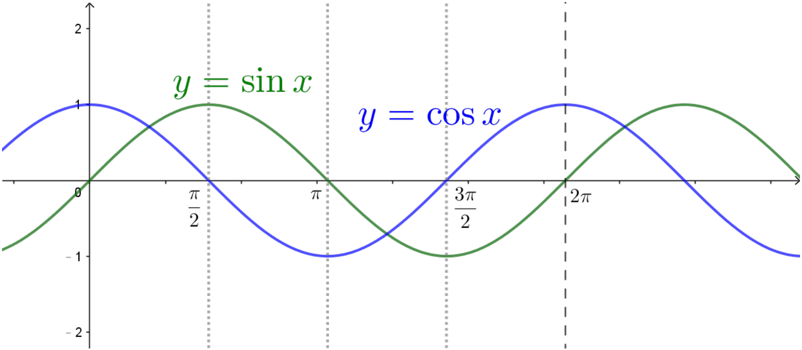

$y = e^x$
$y = \ln x$
4
2
−2
−2
0
2
4
6

Definition (Cosine and sine functions)

The **cosine** and **sine** functions are defined for real number $x \in \mathbb{R}$ by the infinite series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

1. When the sine and cosine are interpreted as trigonometric ratios, the angles are measured in radian. $(180^0 = \pi)$
2. The series for cosine and sine are convergent for any real number $x \in \mathbb{R}$.


$y = \sin x$
$y = \cos x$
2
1
0
-1
-2
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

There are four more trigonometric functions namely tangent, cotangent, secant and cosecant functions. All of them are defined in terms of sine and cosine.

Definition (Trigonometric functions)

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, \ k \in \mathbb{Z} \\
\cot x &= \frac{\cos x}{\sin x}, \text{ for } x \neq k\pi, \ k \in \mathbb{Z} \\
\sec x &= \frac{1}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, \ k \in \mathbb{Z} \\
\csc x &= \frac{1}{\sin x}, \text{ for } x \neq k\pi, \ k \in \mathbb{Z}
\end{aligned}$$

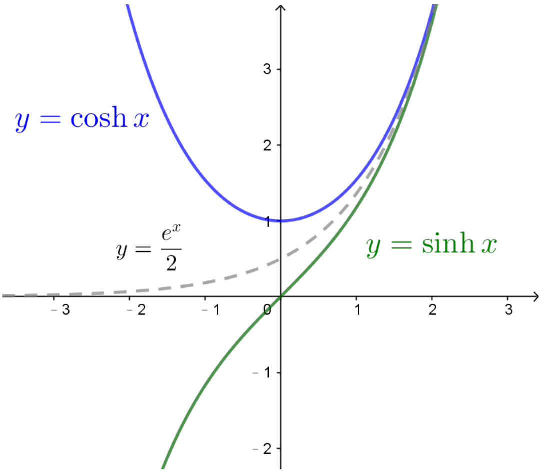
Theorem (Trigonometric identities)

1. $\cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$
2. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y;$
   $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y;$
   $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x;$
   $\sin 2x = 2 \sin x \cos x;$
   $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$
4. $2 \cos x \cos y = \cos(x + y) + \cos(x - y)$
   $2 \cos x \sin y = \sin(x + y) - \sin(x - y)$
   $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$
5. $\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
   $\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$
   $\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$
   $\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$

Definition (Hyperbolic function)

The **hyperbolic functions** are defined for $x \in \mathbb{R}$ by

$$\begin{aligned} \cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \\ \sinh x &= \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \end{aligned}$$


$y = \cosh x$
$y = \frac{e^x}{2}$
$y = \sinh x$
3
2
1
-1
-2
-3
-2
-1
0
1
2
3

Theorem (Hyperbolic identities)

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
   $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
3. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x;$
   $\sinh 2x = 2 \sinh x \cosh x$

Theorem

1. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
2. $\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$
3. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof. $\lim_{x\to 0}\frac{e^x-1}{x}=1.$

For any $-1<x<1$ with $x\neq 0$, we have

$$\begin{aligned}
\frac{e^x-1}{x} &= 1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\frac{x^4}{5!}+\cdots \\
&\le 1+\frac{x}{2}+\left(\frac{x^2}{4}+\frac{x^2}{8}+\frac{x^2}{16}+\cdots\right)=1+\frac{x}{2}+\frac{x^2}{2} \\
\frac{e^x-1}{x} &= 1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots \\
&\ge 1+\frac{x}{2}-\left(\frac{x^2}{4}+\frac{x^2}{8}+\frac{x^2}{16}+\cdots\right)=1+\frac{x}{2}-\frac{x^2}{2}
\end{aligned}$$

and $\lim_{x\to 0}(1+\frac{x}{2}+\frac{x^2}{2})=\lim_{x\to 0}(1+\frac{x}{2}-\frac{x^2}{2})=1$. Therefore $\lim_{x\to 0}\frac{e^x-1}{x}=1$. □

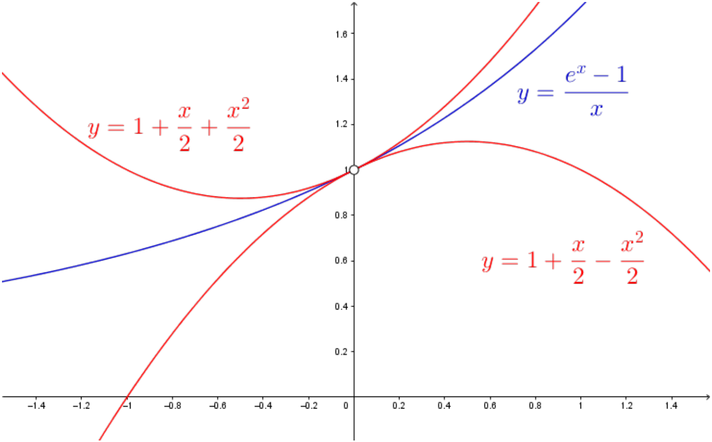


Figure: $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Proof. $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1.$

Let $y = \ln(1+x)$. Then

$$\begin{aligned} e^y &= 1 + x \\ x &= e^y - 1 \end{aligned}$$

and $x \to 0$ as $y \to 0$. We have

$$\begin{aligned} \lim_{x \to 0} \frac{\ln(1+x)}{x} &= \lim_{y \to 0} \frac{y}{e^y - 1} \\ &= 1 \end{aligned}$$

□

Note that the first part implies $\lim_{y \to 0} (e^y - 1) = 0$.

Proof. $\lim_{x\to 0}\frac{\sin x}{x}=1.$

Note that

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}+\cdots.$$

For any $-1<x<1$ with $x\neq 0$, we have

$$\begin{aligned}\frac{\sin x}{x} &= 1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}\right)-\left(\frac{x^6}{7!}-\frac{x^8}{9!}\right)-\cdots\leq 1\\ \frac{\sin x}{x} &= 1-\frac{x^2}{6}+\left(\frac{x^4}{5!}-\frac{x^6}{7!}\right)+\left(\frac{x^8}{9!}-\frac{x^{10}}{11!}\right)+\cdots\geq 1-\frac{x^2}{6}\end{aligned}$$

and $\lim_{x\to 0}1=\lim_{x\to 0}(1-\frac{x^2}{6})=1$. Therefore

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

□

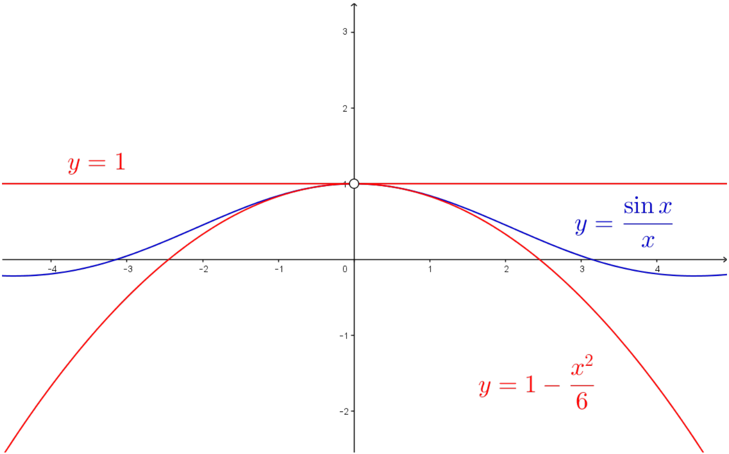


Figure: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Theorem

Let k be a positive integer.

1. $$\lim_{x\to+\infty}\frac{x^k}{e^x}=0$$
2. $$\lim_{x\to+\infty}\frac{(\ln x)^k}{x}=0$$

Proof.

1. For any $x > 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots > \frac{x^{k+1}}{(k+1)!}$$

and thus

$$0 < \frac{x^k}{e^x} < \frac{(k+1)!}{x}.$$

Moreover $\lim_{x\to+\infty} \frac{(k+1)!}{x} = 0$. Therefore

$$\lim_{x\to+\infty} \frac{x^k}{e^x} = 0.$$

2. Let $x = e^y$. Then $x \to +\infty$ as $y \to +\infty$ and $\ln x = y$. We have

$$\lim_{x\to+\infty} \frac{(\ln x)^k}{x} = \lim_{y\to+\infty} \frac{y^k}{e^y} = 0.$$

□

Example

1. $$\lim_{x\to 4}\frac{x^2-16}{\sqrt{x}-2} = \lim_{x\to 4}\frac{(x-4)(x+4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4}\frac{(x-4)(x+4)(\sqrt{x}+2)}{x-4} = \lim_{x\to 4}(x+4)(\sqrt{x}+2) = 32$$

2. $$\lim_{x\to+\infty}\frac{3e^{2x}+e^x-x^4}{4e^{2x}-5e^x+2x^4} = \lim_{x\to+\infty}\frac{3+e^{-x}-x^4e^{-2x}}{4-5e^{-x}+2x^4e^{-2x}} = \frac{3}{4}$$

3. $$\lim_{x\to+\infty}\frac{\ln(2e^{4x}+x^3)}{\ln(3e^{2x}+4x^5)} = \lim_{x\to+\infty}\frac{4x+\ln(2+x^3e^{-4x})}{2x+\ln(3+4x^5e^{-2x})} = \lim_{x\to+\infty}\frac{4+\frac{\ln(2+x^3e^{-4x})}{x}}{2+\frac{\ln(3+4x^5e^{-2x})}{x}} = 2$$

4. $$\lim_{x\to-\infty}(x+\sqrt{x^2-2x}) = \lim_{x\to-\infty}\frac{(x+\sqrt{x^2-2x})(x-\sqrt{x^2-2x})}{x-\sqrt{x^2-2x}} = \lim_{x\to-\infty}\frac{2x}{x-\sqrt{x^2-2x}} = \lim_{x\to-\infty}\frac{2}{1+\sqrt{1-\frac{2}{x}}} = 1$$

Example

5. $$\lim_{x\to 0}\frac{\sin 6x-\sin x}{\sin 4x-\sin 3x} = \lim_{x\to 0}\frac{\frac{6\sin 6x}{6x}-\frac{\sin x}{x}}{\frac{4\sin 4x}{4x}-\frac{3\sin 3x}{3x}} = \frac{6-1}{4-3} = 5$$

6. $$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x\tan x} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x\frac{\sin x}{\cos x}(1+\cos x)} \\ &= \lim_{x\to 0}\frac{(1-\cos^2 x)\cos x}{x\sin x(1+\cos x)} \\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{\cos x}{1+\cos x} = \frac{1}{2}\end{aligned}$$

7. $$\lim_{x\to 0}\frac{e^{2x}-1}{\ln(1+3x)} = \lim_{x\to 0}\frac{2}{3}\cdot\frac{e^{2x}-1}{2x}\cdot\frac{3x}{\ln(1+3x)} = \frac{2}{3}$$

8. $$\begin{aligned}\lim_{x\to 0}\frac{x\ln(1+\sin x)}{1-\sqrt{\cos x}} &= \lim_{x\to 0}\frac{x(1+\sqrt{\cos x})(1+\cos x)\ln(1+\sin x)}{1-\cos^2 x} \\ &= \lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{\ln(1+\sin x)}{\sin x}(1+\sqrt{\cos x})(1+\cos x) \\ &= 4\end{aligned}$$

Continuity of functions

Definition (Continuity)

Let $f(x)$ be a real valued function. We say that $f(x)$ is **continuous** at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

In other words, $f(x)$ is continuous at $x = a$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

We say that $f(x)$ is continuous on an interval in $\mathbb{R}$ if $f(x)$ is continuous at every point on the interval.

Theorem

Let $g(u)$ be a function in u and $u = f(x)$ be a function in x. Suppose $g(u)$ is continuous and the limit of $f(x)$ at $x = a$ exists. Then

$$\lim_{x \to a} (g \circ f)(x) = \lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

$$x \xrightarrow{f} u = f(x) \xrightarrow{g} (g \circ f)(x) = g(u) = g(f(x))$$

Theorem

1. *For any non-negative integer* n, $f(x) = x^n$ *is continuous on* $\mathbb{R}$.
2. *The functions* $e^x, \cos x, \sin x$ *are continuous on* $\mathbb{R}$.
3. *The logarithmic function* $\ln x$ *is continuous on* $\mathbb{R}^+$.

Theorem

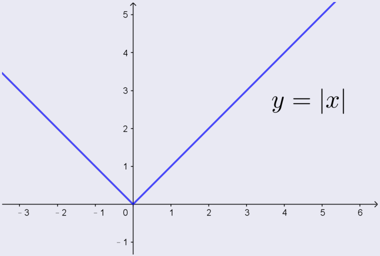
Suppose $f(x)$, $g(x)$ are continuous functions and c is a real number. Then the following functions are continuous.

1. $f(x) + g(x)$
2. $cf(x)$
3. $f(x)g(x)$
4. $\dfrac{f(x)}{g(x)}$ *at the points where* $g(x) \neq 0$.
5. $(f \circ g)(x)$

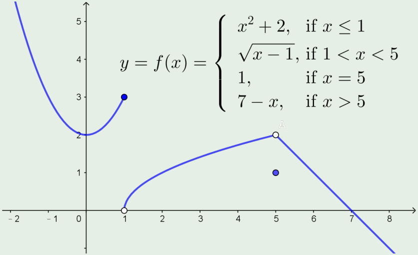
Definition

The **absolute value** of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$



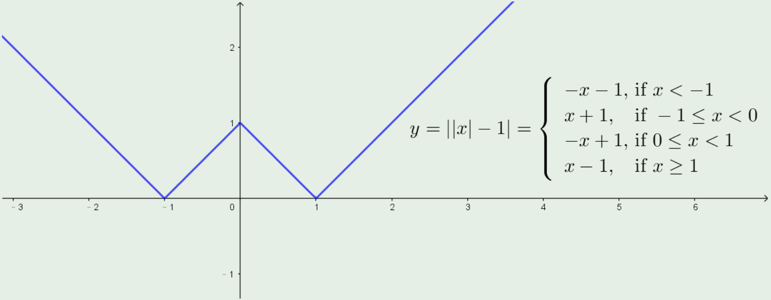
Example (Piecewise defined function)



$$y = f(x) = \begin{cases} x^2+2, & \text{if } x \le 1 \\ \sqrt{x-1}, & \text{if } 1 < x < 5 \\ 1, & \text{if } x = 5 \\ 7-x, & \text{if } x > 5 \end{cases}$$

a	1	5
$\lim_{x\to a^-} f(x)$	3	2
$\lim_{x\to a^+} f(x)$	0	2
$\lim_{x\to a} f(x)$	does not exist	2

Example



$$y = ||x| - 1| = \begin{cases} -x-1, \text{ if } x < -1 \\ x+1, \quad \text{if } -1 \le x < 0 \\ -x+1, \text{ if } 0 \le x < 1 \\ x-1, \quad \text{if } x \ge 1 \end{cases}$$

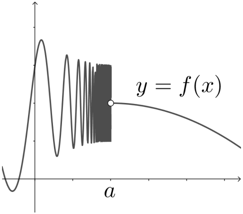
Theorem

A function $f(x)$ *is continuous at* $x = a$ *if*

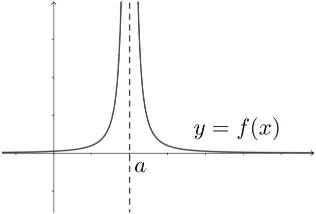
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a).$$

The theorem is usually used to check whether a piecewise defined function is continuous.

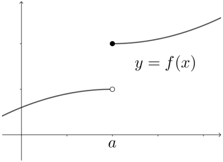
The following functions are not continuous at $x = a$.



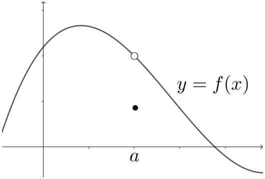
$\lim_{x \to a^-} f(x)$ does not exist



$\lim_{x \to a} f(x)$ does not exist



$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$$



$$\lim_{x \to a} f(x) \neq f(a)$$

Example

Given that the function

$$f(x) = \begin{cases} 2x-1 & \text{if } x<2 \\ a & \text{if } x=2 \\ x^2+b & \text{if } x>2 \end{cases}$$

is continuous at $x = 2$. Find the value of a and b.

Solution

Note that

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}(2x-1) = 3 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+}(x^2+b) = 4+b \\ f(2) &= a \end{aligned}$$

Since $f(x)$ is continuous at $x = 2$, we have $3 = 4 + b = a$ which implies $a = 3$ and $b = -1$.

Example

Prove that the function

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

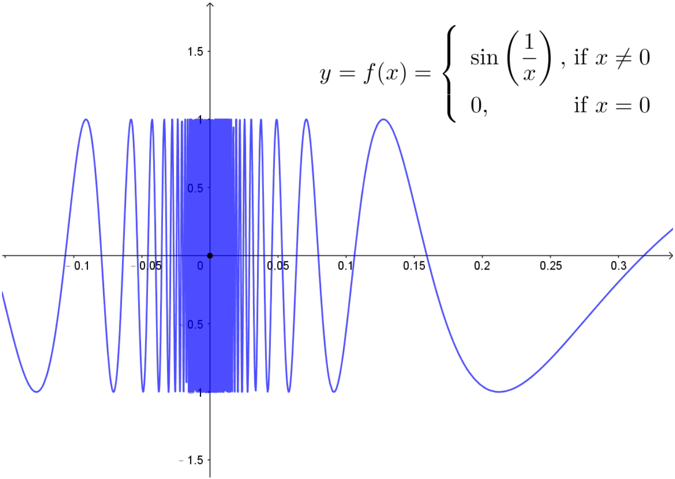
is not continuous at $x = 0$.

Proof.

Let $x_n = \dfrac{2}{(2n+1)\pi}$ for $n = 1, 2, 3, \ldots$. Then $\lim_{n\to\infty} x_n = 0$ and

$$f(x_n) = \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n.$$

Thus $\lim_{n\to\infty} f(x_n)$ does not exist. Therefore $f(x)$ is not continuous at $x = 0$. □

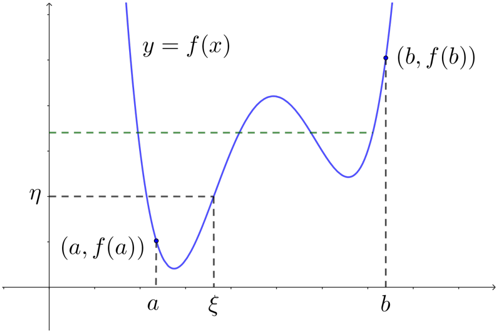


$$y = f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$f(x)$ is not continuous at $x = 0$.

Theorem (Intermediate value theorem)

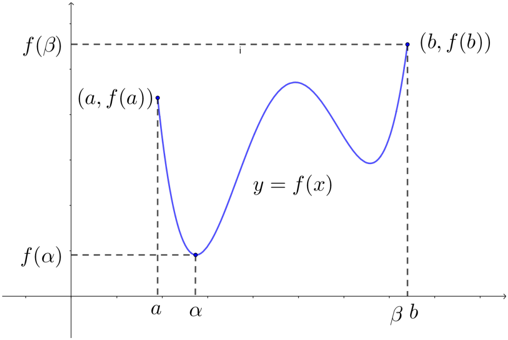
Suppose $f(x)$ is a function which is **continuous** *on $[a, b]$. Then for any real number η between $f(a)$ and $f(b)$, there exists $\xi \in (a, b)$ such that $f(\xi) = \eta$.*



Theorem (Extreme value theorem)

Suppose $f(x)$ *is a function which is* **continuous** *on a* **closed and bounded** *interval* $[a, b]$. *Then there exists* $\alpha, \beta \in [a, b]$ *such that*

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$



Differentiable functions

Definition (Differentiable function)

Let $f(x)$ be a function. Denote

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and we say that $f(x)$ is **differentiable** at $x = a$ if the above limit exists. We say that $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable at every point in (a, b).

The above limit can also be written as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

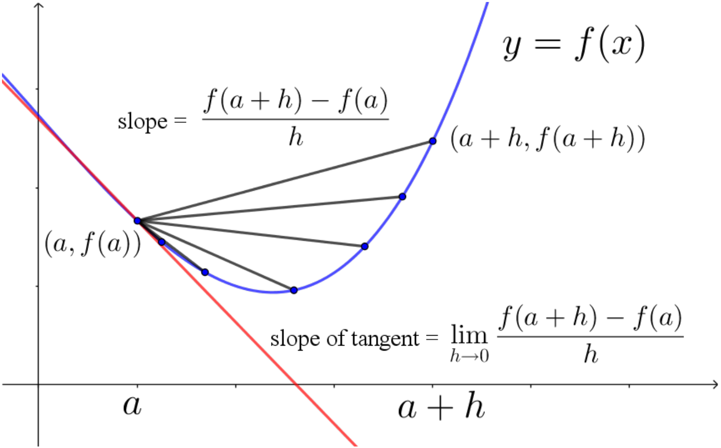
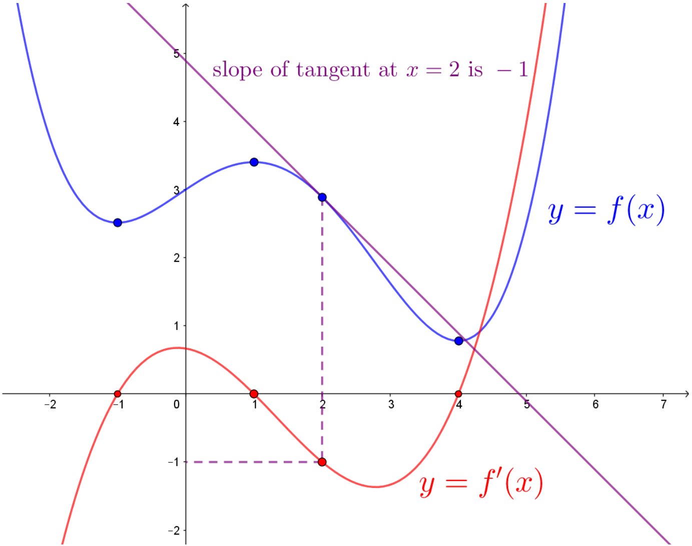


Figure: Definition of derivative


slope of tangent at $x = 2$ is -1
$y = f(x)$
$y = f'(x)$
5
4
3
2
1
−1
−2
−2
−1
0
1
2
3
4
5
6
7

Theorem

If $f(x)$ differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

Differentiable at $x = a \Rightarrow$ **Continuous at** $x = a$

Proof.

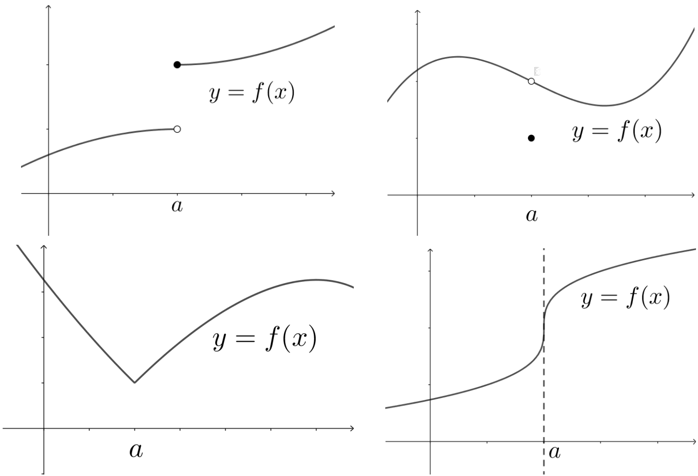
Suppose $f(x)$ is differentiable at $x = a$. Then

$$\begin{aligned}\lim_{x \to a}(f(x) - f(a)) &= \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)(x - a) \\ &= \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)\lim_{x \to a}(x - a) \\ &= f'(a) \cdot 0 = 0\end{aligned}$$

Therefore $f(x)$ is continuous at $x = a$. □

Note that the converse of the above theorem does not hold. The function $f(x) = |x|$ is continuous but not differentiable at 0.

The following functions are not differentiable at $x = a$.



Example

1. $f(x) = e^x$: $f'(0) = \lim_{h\to 0} \dfrac{e^h - e^0}{h} = \lim_{h\to 0} \dfrac{e^h - 1}{h} = 1.$
2. $f(x) = \ln x$: $f'(1) = \lim_{h\to 0} \dfrac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \dfrac{\ln(1+h)}{h} = 1.$
3. $f(x) = \sin x$: $f'(0) = \lim_{h\to 0} \dfrac{\sin h - \sin 0}{h} = \lim_{h\to 0} \dfrac{\sin h}{h} = 1.$

Example

Find the values of a, b if $f(x) = \begin{cases} 4x-1, & \text{if } x \leq 1 \\ ax^2+bx, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Solution: Since $f(x)$ is differentiable at $x = 1$, $f(x)$ is continuous at $x = 1$ and we have

$$\lim_{x \to 1^+} f(x) = f(1) \Rightarrow \lim_{x \to 1^+} (ax^2 + bx) = a + b = 3.$$

Moreover, $f(x)$ is differentiable at $x = 1$ and we have

$$\begin{aligned} \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^-} \frac{(4(1+h) - 1) - 3}{h} = 4 \\ \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^+} \frac{a(1+h)^2 - b(1+h) - 3}{h} = 2a + b \end{aligned}$$

Therefore $\begin{cases} a + b = 3 \\ 2a + b = 4 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$.

Definition (First derivative)

Let $y = f(x)$ be a differentiable function on (a, b). The **first derivative** of $f(x)$ is the function on (a, b) defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Theorem

Let $f(x)$ and $g(x)$ be differentiable functions and c be a real number. Then

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(cf)'(x) = cf'(x)$

Theorem

1. $\dfrac{d}{dx}x^n = nx^{n-1}$, $n \in \mathbb{Z}^+$, *for* $x \in \mathbb{R}$
2. $\dfrac{d}{dx}e^x = e^x$ *for* $x \in \mathbb{R}$
3. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ *for* $x > 0$
4. $\dfrac{d}{dx}\cos x = -\sin x$ *for* $x \in \mathbb{R}$
5. $\dfrac{d}{dx}\sin x = \cos x$ *for* $x \in \mathbb{R}$

Proof ($\dfrac{d}{dx}x^n = nx^{n-1}$)

Let $y = x^n$. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{(x+h-x)((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1})}{h} \\
&= \lim_{h\to 0}((x+h)^{n-1} + (x+h)^{n-2}x + \cdots + x^{n-1}) \\
&= nx^{n-1}
\end{aligned}$$

Note that the above proof is valid only when $n \in \mathbb{Z}^+$ is a positive integer.

Proof ($\frac{d}{dx}e^x = e^x$)

Let $y = e^x$. For any $x \in \mathbb{R}$, we have

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x.$$

(Alternative proof)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) \\ &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= e^x \end{aligned}$$

In general, differentiation cannot be applied term by term to infinite series. The second proof is valid only after we prove that this can be done to **power series**.

Proof

$\left(\frac{d}{dx}\ln x = \frac{1}{x}\right)$ *Let* $f(x) = \ln x$. *For any* $x > 0$, *we have*

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{h} = \frac{1}{x}.$$

$\left(\frac{d}{dx}\cos x = -\sin x\right)$ *Let* $f(x) = \cos x$. *For any* $x \in \mathbb{R}$, *we have*

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = -\sin x.$$

$\left(\frac{d}{dx}\sin x = \cos x\right)$ *Let* $f(x) = \sin x$. *For any* $x \in \mathbb{R}$, *we have*

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} = \cos x.$$

Definition

Let $a > 0$ be a positive real number. For $x \in \mathbb{R}$, we define

$$a^x = e^{x \ln a}.$$

Theorem

Let $a > 0$ be a positive real number. We have

1. $a^{x+y} = a^x a^y$ *for any* $x, y \in \mathbb{R}$
2. $\dfrac{d}{dx} a^x = a^x \ln a.$

Proof.

1. $a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$
2. $\dfrac{d}{dx} a^x = \dfrac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$

□

Example

Let $f(x) = |x|$ for $x \in \mathbb{R}$. Show that $f(x)$ is not differentiable at $x = 0$.

Proof.

Observe that

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{-h}{h} = -1 \\ \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{h}{h} = 1 \end{aligned}$$

Thus the limit

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$. □

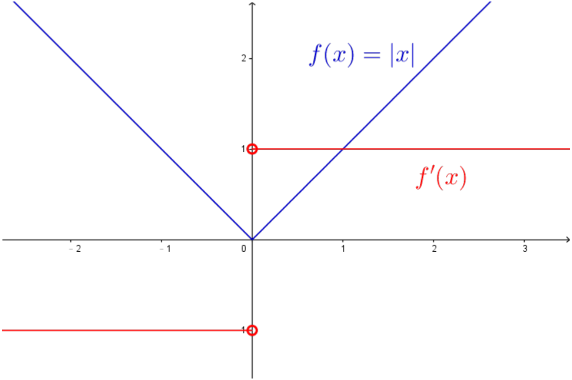


Figure: $f(x) = |x|$ is not differentiable at $x = 0$

Exercise (True or False)

Suppose $f(x)$ is bounded and is differentiable on (a, b). Then

1. *$f'(x)$ is differentiable on (a, b).*
   Answer: **F**
2. *$f'(x)$ is continuous on (a, b).*
   Answer: **F**
3. *$f'(x)$ is bounded on (a, b).*
   Answer: **F**

Example

Let $f(x) = |x|x$ for $x \in \mathbb{R}$. Find $f'(x)$.

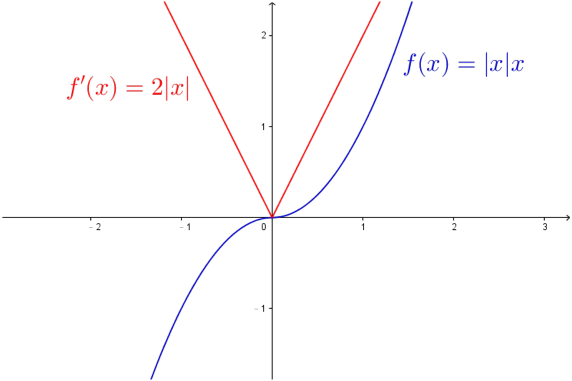
Solution: When $x < 0$, $f(x) = -x^2$ and $f'(x) = -2x$. When $x > 0$, $f(x) = x^2$ and $f'(x) = 2x$. When $x = 0$, we have

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|h - 0}{h} = \lim_{h\to 0} |h| = 0$$

Thus $f'(0) = 0$. Therefore

$$\begin{aligned} f'(x) &= \begin{cases} -2x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 2x, & \text{if } x > 0 \end{cases} \\ &= 2|x|. \end{aligned}$$

Note that $f'(x) = 2|x|$ is continuous at $x = 0$.



- $f(x)$ is differentiable at $x = 0$. ($f(x)$ is differentiable on $\mathbb{R}$.)
- $f'(x)$ is continuous on $\mathbb{R}$.
- $f'(x)$ is not differentiable at $x = 0$.

Example

Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

1. Find $f'(x)$ for $x \neq 0$.
2. Determine whether $f(x)$ is differentiable at $x = 0$.

Solution

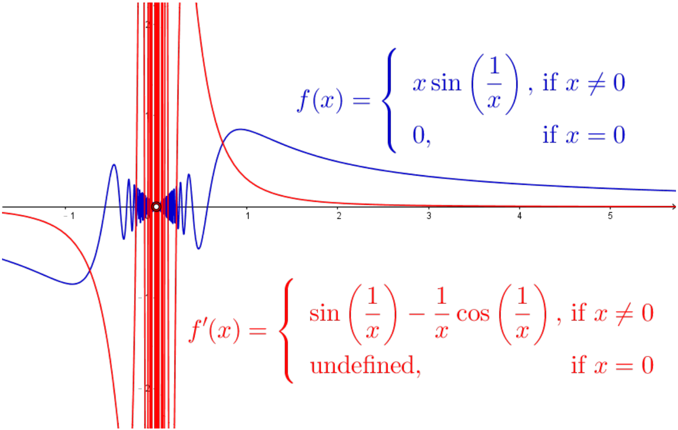
1. When $x \neq 0$,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

2. We have

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h}$$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$.



$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ \text{undefined}, & \text{if } x = 0 \end{cases}$$

- $f(x)$ is not differentiable at $x = 0$.

Example

Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

1. Find $f'(x)$.
2. Determine whether $f'(x)$ is continuous at $x = 0$.

Solution

1. When $x \neq 0$, we have

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left(-\frac{1}{x^2} \cos \frac{1}{x} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Solution

2. *When* $x = 0$, *we have*

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h}.$$

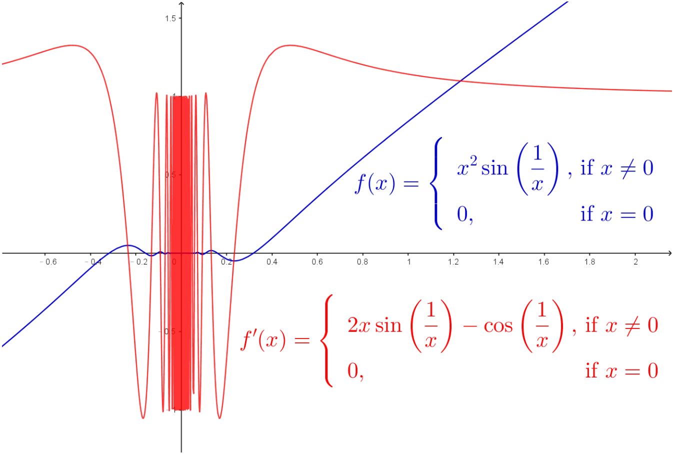
Since $\lim_{h\to 0} h = 0$ *and* $|\sin \frac{1}{h}| \leq 1$ *is bounded, we have* $f'(0) = 0$. *Therefore*

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & \textit{if } x \neq 0 \\ 0, & \textit{if } x = 0 \end{cases}.$$

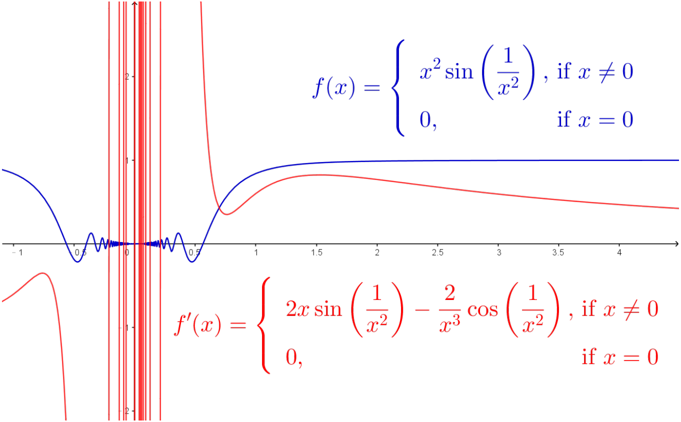
Observe that

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist. We conclude that $f'(x)$ *is not continuous at* $x = 0$.



- $f'(0) = 0$ ($f(x)$ is differentiable on $\mathbb{R}$)
- $f'(x)$ is not continuous at $x = 0$



$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin\left(\dfrac{1}{x^2}\right) - \dfrac{2}{x^3} \cos\left(\dfrac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

- $f'(0) = 0$ ($f(x)$ is differentiable on $\mathbb{R}$)
- $f'(x)$ is not continuous at $x = 0$
- $f'(x)$ is not bounded near $x = 0$

Example

$f(x)$	$f(x)$ is continuous at $x = 0$	$f(x)$ is differentiable at $x = 0$	$f'(x)$ is continuous at $x = 0$
$\lvert x\rvert$	Yes	No	Not applicable
$\lvert x\rvert x$	Yes	Yes	Yes
$x \sin\left(\frac{1}{x}\right);\ f(0) = 0$	Yes	No	Not applicable
$x^2 \sin\left(\frac{1}{x}\right);\ f(0) = 0$	Yes	Yes	No

Example

The following diagram shows the relations between the existence of limit, continuity and differentiability of a function at a point a. (Examples in the bracket is for $a = 0$.)

Second differentiable	$\left(f(x) = \dfrac{\sin x}{x};\ f(0) = 1\right)$
$\Downarrow$	
Continuously differentiable	$(f(x) = \lvert x\rvert x)$
$\Downarrow$	
Differentiable	$(f(x) = x^2 \sin(x^{-1});\ f(0) = 0)$
$\Downarrow$	
Continuous	$(f(x) = \lvert x\rvert)$

Rules of differentiation

Theorem (Basic formulas for differentiation)

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x$$

Theorem (Product rule and quotient rule)

Let u and v be differentiable functions of x. Then

$$\begin{aligned}\frac{d}{dx}uv &= u\frac{dv}{dx}+v\frac{du}{dx}\\ \frac{d}{dx}\frac{u}{v} &= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}\end{aligned}$$

Proof

Let $u = f(x)$ and $v = g(x)$.

$$\begin{aligned}\frac{d}{dx}uv &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\frac{f(x+h)g(x)-f(x)g(x)}{h}\right)\\ &= \lim_{h\to 0}\left(f(x+h)\cdot\frac{g(x+h)-g(x)}{h}+g(x)\cdot\frac{f(x+h)-f(x)}{h}\right)\\ &= u\frac{dv}{dx}+v\frac{du}{dx}\end{aligned}$$

Proof.

$$
\begin{aligned}
\frac{d}{dx}\frac{u}{v} &= \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)g(x)-f(x)g(x)}{hg(x)g(x+h)}-\frac{f(x)g(x+h)-f(x)g(x)}{hg(x)g(x+h)}\right) \\
&= \lim_{h\to 0}\left(g(x)\cdot\frac{f(x+h)-f(x)}{hg(x)g(x+h)}-f(x)\cdot\frac{g(x+h)-g(x)}{hg(x)g(x+h)}\right) \\
&= \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}
\end{aligned}
$$

□

Theorem (Chain rule)

Let $y = f(u)$ be a function of u and $u = g(x)$ be a function of x. Suppose $g(x)$ is differentiable at $x = a$ and $f(u)$ is differentiation at $u = g(a)$. Then $f \circ g(x) = f(g(x))$ is differentiable at $x = a$ and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

In other words,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Proof

$$
\begin{aligned}
& (f\circ g)'(a) \\
= \quad & \lim_{h\to 0}\frac{f(g(a+h))-f(g(a))}{h} \\
= \quad & \lim_{h\to 0}\frac{f(g(a+h))-f(g(a))}{g(a+h)-g(a)}\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} \\
= \quad & \lim_{k\to 0}\frac{f(g(a)+k)-f(g(a))}{k}\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} \\
& (\textit{Note that } g(a+h)-g(a)=k\to 0 \textit{ as } h\to 0 \textit{ because } g(x) \textit{ is continuous.}) \\
= \quad & f'(g(a))g'(a)
\end{aligned}
$$

The above proof is valid only if $g(a+h)-g(a)\neq 0$ whenever h is sufficiently close to 0. This is true when $g'(a)\neq 0$ because of the following proposition.

Proposition

Suppose $g(x)$ is a function such that $g'(a)\neq 0$. Then there exists $\delta>0$ such that if $0<|h|<\delta$, then

$$g(a+h)-g(a)\neq 0.$$

When $g'(a) = 0$, we need another proposition.

Proposition

Suppose $f(u)$ is a function which is differentiable at $u = b$. Then there exists $\delta > 0$ and $M > 0$ such that

$$|f(b+h) - f(b)| < M|h| \text{ for any } |h| < \delta.$$

The proof of chain rule when $g'(a) = 0$ goes as follows. There exists $\delta > 0$ such that

$$|f(g(a+h)) - f(g(a))| < M|g(a+h) - g(a)| \text{ for any } |h| < \delta.$$

Therefore

$$\lim_{h\to 0} \left| \frac{f(g(a+h)) - f(g(a))}{h} \right| \le \lim_{h\to 0} M \left| \frac{g(a+h) - g(a)}{h} \right| = 0$$

which implies $(f \circ g)'(a) = 0$.

Example

The chain rule is used in the following way. Suppose u is a differentiable function of x. Then

$$\begin{aligned}
\frac{d}{dx}u^n &= nu^{n-1}\frac{du}{dx} \\
\frac{d}{dx}e^u &= e^u\frac{du}{dx} \\
\frac{d}{dx}\ln u &= \frac{1}{u}\frac{du}{dx} \\
\frac{d}{dx}\cos u &= -\sin u\frac{du}{dx} \\
\frac{d}{dx}\sin u &= \cos u\frac{du}{dx}
\end{aligned}$$

Example

1. $\dfrac{d}{dx}\sin^3 x = 3\sin^2 x\dfrac{d}{dx}\sin x = 3\sin^2 x\cos x$

2. $\dfrac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}}\dfrac{d}{dx}\sqrt{x} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

3. $\dfrac{d}{dx}\dfrac{1}{(\ln x)^2} = -\dfrac{2}{(\ln x)^3}\dfrac{d}{dx}\ln x = -\dfrac{2}{x(\ln x)^3}$

4. $\dfrac{d}{dx}\ln\cos 2x = \dfrac{1}{\cos 2x}(-\sin 2x)\cdot 2 = -\dfrac{2\sin 2x}{\cos 2x} = -2\tan 2x$

5. $\dfrac{d}{dx}\tan\sqrt{1+x^2} = \sec^2\sqrt{1+x^2}\cdot\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x = \dfrac{x\sec^2\sqrt{1+x^2}}{\sqrt{1+x^2}}$

6. $\dfrac{d}{dx}\sec^3\sqrt{\sin x} = 3\sec^2\sqrt{\sin x}(\sec\sqrt{\sin x}\tan\sqrt{\sin x})\dfrac{1}{2\sqrt{\sin x}}\cdot\cos x$

$$= \frac{3\sec^3\sqrt{\sin x}\tan\sqrt{\sin x}\cos x}{2\sqrt{\sin x}}$$

Example

7. $$\frac{d}{dx}\cos^4 x \sin x = \cos^4 x \cos x + 4\cos^3 x(-\sin x)\sin x = \cos^5 x - 4\cos^3 x \sin^2 x$$

8. $$\frac{d}{dx}\frac{\sec 2x}{\ln x} = \frac{\ln x(2\sec 2x \tan 2x) - \sec 2x(\frac{1}{x})}{(\ln x)^2} = \frac{\sec 2x(2x\tan 2x \ln x - 1)}{x(\ln x)^2}$$

9. $$e^{\frac{\tan x}{x}} = e^{\frac{\tan x}{x}}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)$$

10. $$\sin\left(\frac{\ln x}{\sqrt{1+x^2}}\right) = \cos\left(\frac{\ln x}{\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}(\frac{1}{x}) - \ln x(\frac{2x}{2\sqrt{1+x^2}})}{1+x^2}\right) = \left(\frac{1+x^2-x^2\ln x}{x(1+x^2)^{\frac{3}{2}}}\right)\cos\left(\frac{\ln x}{\sqrt{1+x^2}}\right)$$

Definition (Implicit functions)

An **implicit function** is an equation of the form $F(x, y) = 0$. An implicit function may not define a function. Sometimes it defines a function when the domain and range are specified.

Theorem

Let $F(x, y) = 0$ be an implicit function. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

and we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Here $\frac{\partial F}{\partial x}$ is called the partial derivative of F with respect to x which is the derivative of F with respect to x while considering y as constant. Similarly the partial derivative $\frac{\partial F}{\partial y}$ is the derivative of F with respect to y while considering x as constant.

Example

Find $\dfrac{dy}{dx}$ for the following implicit functions.

1. $x^2 - xy - xy^2 = 0$
2. $\cos(xe^y) + x^2 \tan y = 1$

Solution

$$\begin{aligned} 1.\ 2x - (y + xy') - (y^2 + 2xyy') &= 0 \\ xy' + 2xyy' &= 2x - y - y^2 \\ y' &= \frac{2x - y - y^2}{x + 2xy} \end{aligned}$$

$$\begin{aligned} 2.\ -\sin(xe^y)(e^y + xe^y y') + 2x\tan y + x^2\sec^2 y\, yy' &= 0 \\ x^2 \sec^2 y\, yy' - xe^y \sin(xe^y) y' &= e^y \sin(xe^y) - 2x\tan y \\ y' &= \frac{e^y \sin(xe^y) - 2x\tan y}{x^2 \sec^2 y - xe^y \sin(xe^y)} \end{aligned}$$

Theorem

Suppose $f(y)$ is a differentiable function with $f'(y) \neq 0$ for any y. Then the inverse function $y = f^{-1}(x)$ of $f(y)$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In other words,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Proof.

$$\begin{aligned} f(f^{-1}(x)) &= x \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

□

Theorem

1. *For* $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

2. *For* $\cos^{-1} : [-1,1] \to [0,\pi]$,

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

3. *For* $\tan^{-1} : \mathbb{R} \to [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

Proof.

1.

$$
\begin{aligned}
y &= \sin^{-1} x \\
\sin y &= x \\
\cos y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin^2 y}} \quad \text{(Note: } \cos y \geq 0 \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\text{)} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

The other parts can be proved similarly. □

Example

Find $\dfrac{dy}{dx}$ if $y = x^x$.

Solution

There are 2 methods.
Method 1. Note that $y = x^x = e^{x\ln x}$. *Thus*

$$\frac{dy}{dx} = e^{x\ln x}(1 + \ln x) = x^x(1 + \ln x).$$

Method 2. Taking logarithm on both sides, we have

$$\begin{aligned} \ln y &= x\ln x \\ \frac{1}{y}\frac{dy}{dx} &= 1 + \ln x \\ \frac{dy}{dx} &= y(1 + \ln x) \\ &= x^x(1 + \ln x) \end{aligned}$$

Example

Let u and v be functions of x. Show that

$$\frac{d}{dx}u^v = u^v v' \ln u + u^{v-1} v u'.$$

Proof.

We have

$$\begin{aligned} \frac{d}{dx}u^v &= \frac{d}{dx}e^{v \ln u} \\ &= e^{v \ln u}\left(v' \ln u + v \cdot \frac{u'}{u}\right) \\ &= u^v \left(v' \ln u + \frac{vu'}{u}\right) \\ &= u^v v' \ln u + u^{v-1} v u' \end{aligned}$$

□

Second and higher derivatives

Definition (Second and higher derivatives)

Let $y = f(x)$ be a function. The **second derivative** of $f(x)$ is the function

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The second derivative of $y = f(x)$ is also denoted as $f''(x)$ or y''. Let n be a non-negative integer. The n**-th derivative** of $y = f(x)$ is defined inductively by

$$\begin{aligned} \frac{d^ny}{dx^n} &= \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \quad \text{for } n \geq 1 \\ \frac{d^0y}{dx^0} &= y \end{aligned}$$

The n-th derivative is also denoted as $f^{(n)}(x)$ or $y^{(n)}$. Note that $f^{(0)}(x) = f(x)$.

Example

Find $\frac{d^2y}{dx^2}$ for the following functions.

1. $y = \ln(\sec x + \tan x)$
2. $x^2 - y^2 = 1$

Solution

$$\begin{aligned} 1.\quad y' &= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) \\ &= \sec x \\ y'' &= \sec x \tan x \end{aligned}$$

$$\begin{aligned} 2.\quad 2x - 2yy' &= 0 \\ y' &= \frac{x}{y} \\ y'' &= \frac{y - xy'}{y^2} \\ &= \frac{y - x(\frac{x}{y})}{y^2} \\ &= \frac{y^2 - x^2}{y^3} \end{aligned}$$

Theorem (Leinbiz's rule)

Let u and v be differentiable function of x. Then

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binormial coefficient.

Example

$$\begin{aligned}
(uv)^{(0)} &= u^{(0)}v^{(0)} \\
(uv)^{(1)} &= u^{(1)}v^{(0)} + u^{(0)}v^{(1)} \\
(uv)^{(2)} &= u^{(2)}v^{(0)} + 2u^{(1)}v^{(1)} + u^{(0)}v^{(2)} \\
(uv)^{(3)} &= u^{(3)}v^{(0)} + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + u^{(0)}v^{(3)} \\
(uv)^{(4)} &= u^{(4)}v^{(0)} + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + u^{(0)}v^{(4)}
\end{aligned}$$

Proof

We prove the Leibniz's rule by induction on n. *When* $n = 0$, $(uv)^{(0)} = uv = u^{(0)}v^{(0)}$. *Assume that for some nonnegative* m,

$$(uv)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)}.$$

Then

$$\begin{aligned}
& (uv)^{(m+1)} \\
= \quad & \frac{d}{dx}(uv)^{(m)} \\
= \quad & \frac{d}{dx} \sum_{k=0}^{m} \binom{m}{k} u^{(m-k)} v^{(k)} \\
= \quad & \sum_{k=0}^{m} \binom{m}{k} (u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)})
\end{aligned}$$

Proof.

$$\begin{aligned}
&= \sum_{k=0}^{m}\binom{m}{k}u^{(m-k+1)}v^{(k)}+\sum_{k=0}^{m}\binom{m}{k}u^{(m-k)}v^{(k+1)} \\
&= \sum_{k=0}^{m}\binom{m}{k}u^{(m-k+1)}v^{(k)}+\sum_{k=1}^{m+1}\binom{m}{k-1}u^{(m-(k-1))}v^{(k)} \\
&= \sum_{k=0}^{m}\binom{m}{k}u^{(m-k+1)}v^{(k)}+\sum_{k=1}^{m+1}\binom{m}{k-1}u^{(m-k+1)}v^{(k)} \\
&= \sum_{k=0}^{m+1}\left(\binom{m}{k}+\binom{m}{k-1}\right)u^{(m-k+1)}v^{(k)} \\
&= \sum_{k=0}^{m+1}\binom{m+1}{k}u^{(m+1-k)}v^{(k)}
\end{aligned}$$

Here we use the convention $\binom{m}{-1}=\binom{m}{m+1}=0$ in the second last equality. This completes the induction step and the proof of the Leibniz's rule. □

Example

Let $y = x^2 e^{3x}$. Find $y^{(n)}$ where n is a nonnegative integer.

Solution

Let $u = x^2$ *and* $v = e^{3x}$. *Then* $u^{(0)} = x^2$, $u^{(1)} = 2x$, $u^{(2)} = 2$ *and* $u^{(k)} = 0$ *for* $k \geq 3$. *On the other hand,* $v^{(k)} = 3^k e^{3x}$ *for any* $k \geq 0$. *Therefore by Leibniz's rule, we have*

$$\begin{aligned}
y^{(n)} &= \binom{n}{0} u^{(0)} v^{(n)} + \binom{n}{1} u^{(1)} v^{(n-1)} + \binom{n}{2} u^{(2)} v^{(n-2)} \\
&= x^2(3^n e^{3x}) + n(2x)(3^{n-1} e^{3x}) + \frac{n(n-1)}{2!}(2)(3^{n-2} e^{3x}) \\
&= (3^n x^2 + 2 \cdot 3^{n-1} nx + 3^{n-2}(n^2 - n))e^{3x} \\
&= 3^{n-2}(9x^2 + 6nx + n^2 - n)e^{3x}
\end{aligned}$$

Mean value theorem

Suppose $f(x)$ is a function which is differentiable on (a, b).

1. If $f(x)$ attains its maximum or minimum at $x = c \in (a, b)$, then $f'(c) = 0$.
   Answer: **T**
2. If $f'(c) = 0$, then $f(x)$ attains its maximum or minimum at $x = c \in (a, b)$.
   Answer: **F**
3. If $f'(x) = 0$ for any $x \in (a, b)$, then $f(x)$ is constant on (a, b).
   Answer: **T**
4. If $f(x)$ is strictly increasing on (a, b), then $f'(x) > 0$ for any $x \in (a, b)$.
   Answer: **F**
5. If $f'(x) > 0$ for any (a, b), then $f(x)$ is strictly increasing on (a, b).
   Answer: **T**
6. If $f(x)$ is monotonic increasing on (a, b), then $f'(x) \geq 0$ for any $x \in (a, b)$.
   Answer: **T**

Theorem

Let f be a function on (a, b) and $c \in (a, b)$ such that

1. *f is differentiable at $x = c$, and*
2. *either $f(x) \leq f(c)$ for any $x \in (a, b)$, or $f(x) \geq f(c)$ for any $x \in (a, b)$.*

Then $f'(c) = 0$.

Proof.

Suppose $f(x) \leq f(c)$ for any $x \in (a, b)$. The proof for the other case is essentially the same. For any $h < 0$ with $a < c + h < c$, we have $f(c + h) - f(c) \leq 0$ and h is negative. Thus

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

On the other hand, for any $h > 0$ with $c < c + h < b$, we have $f(c + h) - f(c) \leq 0$ and h is positive. Thus we have

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

Therefore $f'(c) = 0$. □

Example

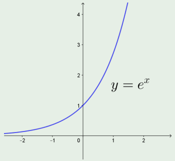
$f'(x) > 0$ for any x

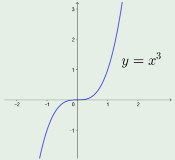
$\Downarrow$

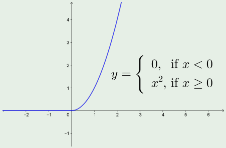
Strictly increasing

$\Downarrow$

Monotonic increasing $\Leftrightarrow$ $f'(x) \geq 0$ for any x





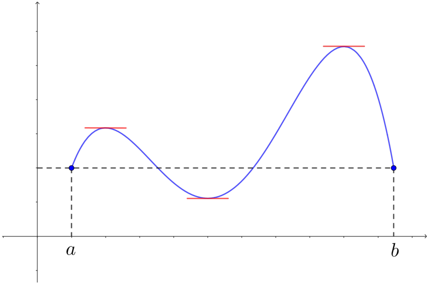


Theorem (Rolle's theorem)

Suppose $f(x)$ *is a function which satisfies the following conditions.*

1. $f(x)$ *is continuous on* $[a, b]$.
2. $f(x)$ *is differentiable on* (a, b).
3. $f(a) = f(b)$

Then there exists $\xi \in (a, b)$ *such that* $f'(\xi) = 0$.



Proof.

By extreme value theorem, there exist $a \leq \alpha, \beta \leq b$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$

Since $f(a) = f(b)$, at least one of α, β can be chosen in (a, b) and we let it be ξ. Then we have $f'(\xi) = 0$ since $f(x)$ attains its maximum or minimum at ξ. □

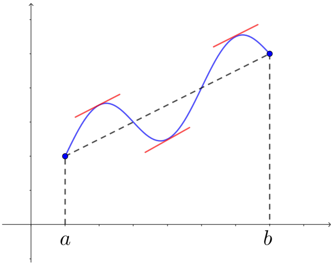
Theorem (Lagrange's mean value theorem)

Suppose $f(x)$ is a function which satisfies the following conditions.

1. $f(x)$ *is continuous on* $[a, b]$.
2. $f(x)$ *is differentiable on* (a, b).

Then there exists $\xi \in (a, b)$ *such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



Proof.

Let $g(x) = f(x) - \dfrac{f(b) - f(a)}{b - a}(x - a)$. Since $g(a) = g(b) = f(a)$, by Rolle's theorem, there exists $\xi \in (a, b)$ such that

$$\begin{aligned} g'(\xi) &= 0 \\ f'(\xi) - \frac{f(b) - f(a)}{b - a} &= 0 \\ f'(\xi) &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

□

Theorem

Let $f(x)$ be a function which is differentiable on (a, b). Then $f(x)$ is monotonic increasing if and only if $f'(x) \geq 0$ for any $x \in (a, b)$.

Proof. Suppose $f(x)$ is monotonic increasing on (a, b). Then for any $x \in (a, b)$, we have $f(x+h) - f(x) \geq 0$ for any $h > 0$ and thus

$$f'(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \geq 0.$$

On the other hand, suppose $f'(x) \geq 0$ for any $x \in (a, b)$. Then for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta$, by Lagrange's mean value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) \geq 0.$$

Therefore $f(x)$ is monotonic increasing on (a, b). □

Corollary

$f(x)$ is constant on (a, b) if and only if $f'(x) = 0$ for any $x \in (a, b)$.

Theorem

If $f(x)$ is a differentiable function such that $f'(x) > 0$ for any $x \in (a, b)$, then $f(x)$ is strictly increasing.

Proof.

Suppose $f'(x) > 0$ for any $x \in (a, b)$. Then for any $\alpha, \beta \in (a, b)$ with $\alpha < \beta$, by Lagrange's mean value theorem, there exists $\xi \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) > 0.$$

Therefore $f(x)$ is strictly increasing on (a, b). □

The converse of the above theorem is false.

Example

$f(x) = x^3$ is strictly increasing on $\mathbb{R}$ but $f'(0) = 0$ is not positive.

Example

Prove that $1 - \frac{1}{x} \leq \ln x \leq x - 1$ for any $x > 0$.

Solution. Let $f(x) = \ln x - \left(1 - \frac{1}{x}\right)$. Then $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$. Now $f'(1) = 0$ and

	$0 < x < 1$	$x > 1$
$f'(x)$	$-$	$+$

Therefore $f(x)$ attains its minimum at $x = 1$ and we have $f(x) = \ln x - \frac{x-1}{x} \geq f(1) = 0$ for any $x > 0$. On the other hand, let $g(x) = x - 1 - \ln x$. Then $g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. Now $g'(1) = 0$ and

	$0 < x < 1$	$x > 1$
$f'(x)$	$-$	$+$

Therefore $g(x)$ attains its minimum at $x = 1$ and we have $g(x) = x - 1 - \ln x \geq g(1) = 0$ for any $x > 0$.

Example

Let $0 < \alpha < 1$. Prove that

$$1 + \alpha x - \frac{\alpha(1-\alpha)x^2}{2} < (1+x)^\alpha < 1 + \alpha x, \text{ for any } x > 0.$$

Solution. Let $f(x) = 1 + \alpha x - (1+x)^\alpha$. Then $f(0) = 0$ and for any $x > 0$,

$$f'(x) = \alpha - \frac{\alpha}{(1+x)^{1-\alpha}} > \alpha - \alpha = 0.$$

Therefore $f(x) > 0$ for any $x > 0$. On the other hand, let $g(x) = (1+x)^\alpha - \left(1 + \alpha x - \frac{\alpha(1-\alpha)x^2}{2}\right)$. Then $g(0) = 0$ and for any $x > 0$,

$$\begin{aligned} g'(x) &= \frac{\alpha}{(1+x)^{1-\alpha}} - \alpha + \alpha(1-\alpha)x \\ &> \frac{\alpha}{1+(1-\alpha)x} - \alpha(1-(1-\alpha)x) \\ &= \frac{\alpha(1-\alpha)^2x^2}{1+(1-\alpha)x} > 0 \end{aligned}$$

Therefore $g(x) > 0$ for any $x > 0$.

Theorem (Cauchy's mean value theorem)

Suppose $f(x)$ *and* $g(x)$ *are functions which satisfies the following conditions.*

1. $f(x), g(x)$ *is continuous on* $[a, b]$.
2. $f(x), g(x)$ *is differentiable on* (a, b).
3. $g'(x) \neq 0$ *for any* $x \in (a, b)$.

Then there exists $\xi \in (a, b)$ *such that*

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Let $h(x) = f(x) - \dfrac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$.

Since $h(a) = h(b) = f(a)$, by Rolle's theorem, there exists $\xi \in (a, b)$ such that

$$\begin{aligned} f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(\xi) &= 0 \\ \frac{f'(\xi)}{g'(\xi)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

L'Hopital's rule

Theorem (L'Hopital's rule)

Let $a \in [-\infty, +\infty]$. Suppose f and g are differentiable functions such that

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *(or $\pm\infty$).*
2. $g'(x) \neq 0$ *for any $x \neq a$ (on a neighborhood of a).*
3. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$

Then the limit of $\frac{f(x)}{g(x)}$ at $x = a$ exists and $\lim_{x\to a} \frac{f(x)}{g(x)} = L.$

Proof.

We give here the proof for $a \in (-\infty, +\infty)$. For any $x \neq a$, by applying Cauchy's mean value theorem to $f(x)$, $g(x)$ on $[a, x]$ or $[x, a]$, there exists ξ between a and x such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Here we redefine $f(a) = g(a) = 0$, if necessary, so that f and g are continuous at a. Note that $\xi \to a$ as $x \to a$. We have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(\xi)}{g'(\xi)} = L.$$

□

Example (Indeterminate form of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$)

1. $\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3} = \lim_{x\to 0}\frac{x\sin x}{3x^2} = \frac{1}{3}$
2. $\lim_{x\to 0}\frac{x^2}{\ln\sec x} = \lim_{x\to 0}\frac{2x}{\frac{\sec x\tan x}{\sec x}} = \lim_{x\to 0}\frac{2x}{\tan x} = \lim_{x\to 0}\frac{2}{\sec^2 x} = 2$
3. $\lim_{x\to 0}\frac{\ln(1+x^3)}{x-\sin x} = \lim_{x\to 0}\frac{\frac{3x^2}{1+x^3}}{1-\cos x} = \lim_{x\to 0}\frac{3}{1+x^3}\lim_{x\to 0}\frac{x^2}{1-\cos x}$
   $= 3\lim_{x\to 0}\frac{2x}{\sin x} = 6$
4. $\lim_{x\to+\infty}\frac{\ln(1+x^4)}{\ln(1+x^2)} = \lim_{x\to+\infty}\frac{\frac{4x^3}{1+x^4}}{\frac{2x}{1+x^2}} = \lim_{x\to+\infty}\frac{4x^3(1+x^2)}{2x(1+x^4)} = 2$

Example (Indeterminate form of types $\infty - \infty$ and $0 \cdot \infty$)

5. $$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1}\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}$$
$$= \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} = \lim_{x\to 1}\frac{1}{2+\ln x} = \frac{1}{2}$$

6. $$\lim_{x\to 0}\cot 3x\tan^{-1}x = \lim_{x\to 0}\frac{\tan^{-1}x}{\tan 3x} = \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{3\sec^2 3x}$$
$$= \lim_{x\to 0}\frac{1}{3(1+x^2)\sec^2 3x} = \frac{1}{3}$$

7. $$\lim_{x\to 0^+} x\ln\sin x = \lim_{x\to 0^+}\frac{\ln\sin x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+}\frac{-x^2\cos x}{\sin x} = 0$$

8. $$\lim_{x\to+\infty} x\ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to+\infty}\frac{\ln(x+1)-\ln(x-1)}{\frac{1}{x}}$$
$$= \lim_{x\to+\infty}\frac{\frac{1}{x+1}-\frac{1}{x-1}}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{2x^2}{(x+1)(x-1)} = 2$$

Example (Indeterminate form of types 0^0, 1^∞ and ∞^0)

Evaluate the following limits.

1. $\displaystyle\lim_{x\to 0^+} x^{\sin x}$
2. $\displaystyle\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$
3. $\displaystyle\lim_{x\to +\infty} (1+2x)^{\frac{1}{3\ln x}}$

Solution

1. $$\ln\left(\lim_{x\to 0^+} x^{\sin x}\right) = \lim_{x\to 0^+}\ln(x^{\sin x}) = \lim_{x\to 0^+}\sin x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\csc x}$$
$$= \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\csc x\cot x} = \lim_{x\to 0^+}\frac{-\sin^2 x}{x\cos x} = 0.$$
Thus $\lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$

2. $$\ln\left(\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}\right) = \lim_{x\to 0}\ln(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\frac{\ln\cos x}{x^2} = \lim_{x\to 0}\frac{-\tan x}{2x}$$
$$= \lim_{x\to 0}\frac{-\sec^2 x}{2} = -\frac{1}{2}.$$
Thus $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}.$

3. $$\ln\left(\lim_{x\to+\infty}(1+2x)^{\frac{3}{\ln x}}\right) = \lim_{x\to+\infty}\frac{3\ln(1+2x)}{\ln x} = \lim_{x\to+\infty}\frac{\frac{6}{1+2x}}{\frac{1}{x}} = 3.$$
Thus $\lim_{x\to+\infty}(1+2x)^{\frac{1}{3\ln x}} = e^3.$

Example

The following shows some wrong use of L'Hopital rule.

1.

$$\begin{aligned}\lim_{x\to 0}\frac{\sec x - 1}{e^{2x}-1} &= \lim_{x\to 0}\frac{\sec x\tan x}{2e^{2x}}\\ &= \lim_{x\to 0}\frac{\sec^2 x\tan x + \sec^3 x}{4e^{2x}}\\ &= \frac{1}{4}\end{aligned}$$

This is wrong because $\lim_{x\to 0} e^{2x} \neq 0, \pm\infty$. One cannot apply L'Hopital rule to $\lim_{x\to 0}\frac{\sec x\tan x}{2e^{2x}}$. The correct solution is

$$\lim_{x\to 0}\frac{\sec x - 1}{e^{2x}-1} = \lim_{x\to 0}\frac{\sec x\tan x}{2e^{2x}} = 0.$$

Example

2.

$$\begin{aligned}\lim_{x\to+\infty}\frac{5x-2\cos^2 x}{3x+\sin^2 x} &= \lim_{x\to+\infty}\frac{5+2\cos x\sin x}{3+\sin x\cos x}\\ &= \lim_{x\to+\infty}\frac{2(\cos^2 x-\sin^2 x)}{\cos^2 x-\sin^2 x}\\ &= 2\end{aligned}$$

This is wrong because $\lim_{x\to+\infty}(5+2\cos x\sin x)$ and $\lim_{x\to+\infty}(3+\cos x\sin x)$ do not exist. One cannot apply L'Hopital rule to $\lim_{x\to+\infty}\frac{5+2\cos x\sin x}{3+\sin x\cos x}$. The correct solution is

$$\lim_{x\to+\infty}\frac{5x-2\cos^2 x}{3x+\sin^2 x}=\lim_{x\to+\infty}\frac{5-\frac{2\cos^2 x}{x}}{3+\frac{\sin^2 x}{x}}=\frac{5}{3}.$$

Taylor series

Definition (Taylor polynomial)

Let $f(x)$ be a function such that the n-th derivative exists at $x = a$. The **Taylor polynomial** of degree n of $f(x)$ at $x = a$ is the polynomial

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem

The Taylor polynomial $p_n(x)$ of degree n of $f(x)$ at $x = a$ is the unique polynomial such that

$$p_n^{(k)}(a) = f^{(k)}(a) \textit{ for } k = 0, 1, 2, \ldots, n.$$

Example

Find the Taylor polynomial $p_3(x)$ of degree 3 of $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$ at $x = 0$.

Solution. The derivatives $f^{(k)}(x)$ up to order 3 are

k	0	1	2	3
$f^{(k)}(x)$	$(1+x)^{\frac{1}{2}}$	$\frac{1}{2}(1+x)^{-\frac{1}{2}}$	$-\frac{1}{4}(1+x)^{-\frac{3}{2}}$	$\frac{3}{8}(1+x)^{-\frac{5}{2}}$
$f^{(k)}(0)$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$

Therefore the Taylor polynomial of $f(x)$ of degree 3 at $x = 0$ is

$$\begin{aligned} p_3(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} \\ &= 1 + \left(\frac{1}{2}\right)x + \left(-\frac{1}{4}\right)\frac{x^2}{2!} + \left(\frac{3}{8}\right)\frac{x^3}{3!} \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \end{aligned}$$

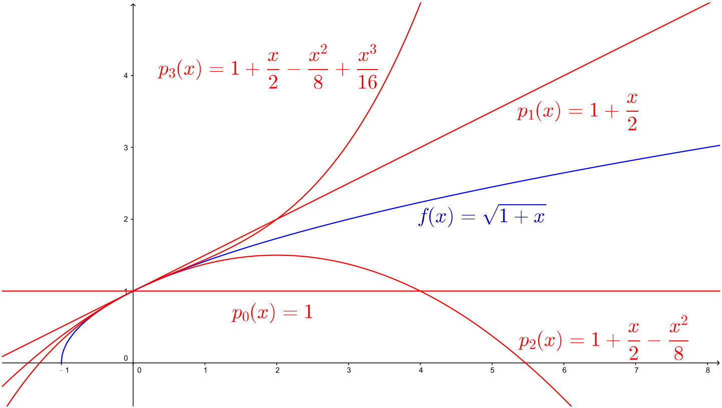


Figure: Taylor polynomials for $f(x) = \sqrt{1+x}$ at $x = 0$

Example

Let $f(x) = \cos x$. The first few derivatives are

k	0	1	2	3	4
$f^{(k)}(x)$	$\cos x$	$-\sin x$	$-\cos x$	$-\sin x$	$\cos x$
$f^{(k)}(0)$	1	0	-1	0	1

We see that

$$f^{(n)}(x) = \begin{cases} (-1)^k \cos x, & \text{if } n = 2k \\ (-1)^k \sin x, & \text{if } n = 2k-1 \end{cases} \text{ and } f^{(n)}(0) = \begin{cases} (-1)^k, & \text{if } n = 2k \\ 0, & \text{if } n = 2k-1 \end{cases}$$

Therefore the Taylor polynomial of $f(x)$ of degree $n = 2k$ at $x = 0$ is

$$\begin{aligned} p_{2k}(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(2k)x^{2k}}(0)}{(2k)!} \\ &= 1 + (0)x + \frac{(-1)x^2}{2!} + \frac{(0)x^3}{3!} + \frac{(1)x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

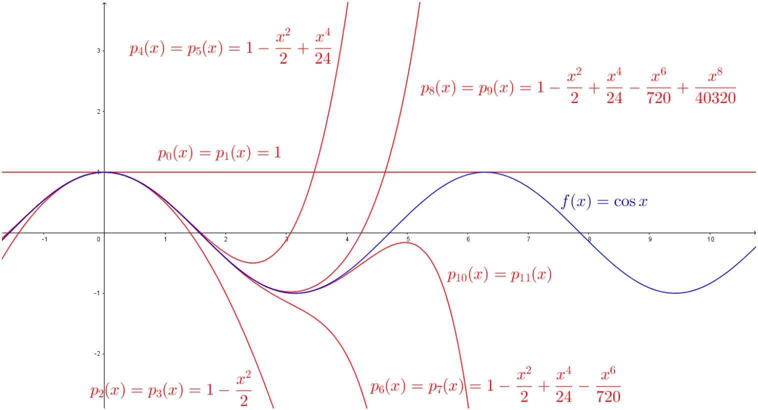


Figure: Taylor polynomials for $f(x) = \cos x$ at $x = 0$

Example

Find the Taylor polynomial of degree n of $f(x) = \dfrac{1}{x}$ at $x = 1$.

Solution. The derivatives $f^{(k)}(x)$ are

k	0	1	2	3	$\cdots$	n
$f^{(k)}(x)$	x^{-1}	$-x^{-2}$	$2x^{-3}$	$-6x^{-4}$	$\cdots$	$(-1)^n n! x^{-(n+1)}$
$f^{(k)}(1)$	1	-1	2	-6	$\cdots$	$(-1)^n n!$

Therefore the Taylor polynomial of $f(x)$ of degree n at $x = 1$ is

$$\begin{aligned} p_n(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \cdots + \frac{f^{(n)}(1)(x-1)^n}{(n)!} \\ &= 1 - (x-1) + \frac{2(x-1)^2}{2!} + \frac{(-6)(x-1)^3}{3!} + \cdots + \frac{(-1)^n n!(x-1)^n}{n!} \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n \end{aligned}$$

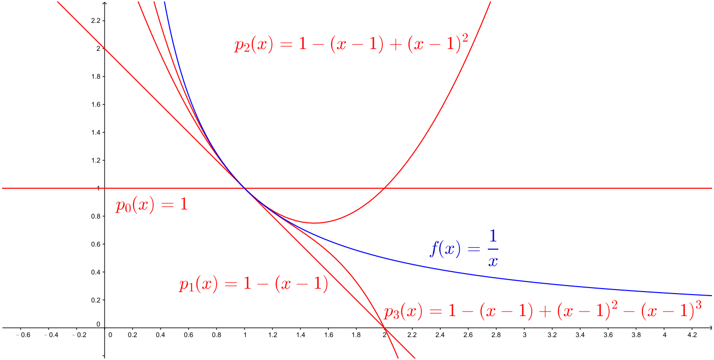


Figure: Taylor polynomials for $f(x) = \dfrac{1}{x}$ at $x = 1$

Example

Find the Taylor polynomial of $f(x) = (1+x)^{\alpha}$ at $x = 0$, where $\alpha \in \mathbb{R}$.
Solution. The derivatives are

$$\begin{aligned}
f(x) &= (1+x)^{\alpha} \\
f'(x) &= \alpha(1+x)^{\alpha-1} \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} \\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \\
&\vdots \\
f^{(k)}(x) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}
\end{aligned}$$

Example

Thus we have

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= \alpha \\ f''(0) &= \alpha(\alpha-1) \\ &\vdots \\ f^{(k)}(0) &= \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1) \end{aligned}$$

Therefore the Taylor polynomial of $f(x) = (1+x)^\alpha$ of degree n at $x = 0$ is

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{(n)!} \\ &= 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)x^n}{n!} \\ &= \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n \end{aligned}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

Example

The Taylor polynomials of degree n for $f(x)$ at $x=0$.

$f(x)$	Taylor polynomial
e^x	$1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^n}{n!}$
$\cos x$	$1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots+\dfrac{(-1)^k x^{2k}}{(2k)!},\ n=2k$
$\sin x$	$x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots+\dfrac{(-1)^k x^{2k+1}}{(2k+1)!},\ n=2k+1$
$\ln(1+x)$	$x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots+\dfrac{(-1)^{n+1}x^n}{n}$
$\dfrac{1}{1-x}$	$1+x+x^2+x^3+\cdots+x^n$
$\sqrt{1+x}$	$1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}+\cdots+\dfrac{(-1)^{n+1}(2n-3)!!x^n}{2^n n!}$
$(1+x)^\alpha$	$1+\alpha x+\dfrac{\alpha(\alpha-1)x^2}{2!}+\dfrac{\alpha(\alpha-1)(\alpha-2)x^3}{3!}+\cdots+\dbinom{\alpha}{n}x^n$

Example

The Taylor polynomials of degree n for $f(x)$ at $x = a$.

$f(x)$	Taylor polynomial
$\cos x;\ a = \pi$	$-1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \cdots + \frac{(-1)^{k+1}(x-\pi)^{2k}}{(2k)!}$
$e^x;\ a = 2$	$e^2 + e^2(x-2) + \frac{e^2(x-2)^2}{2!} + \cdots + \frac{e^2(x-2)^n}{n!}$
$\frac{1}{x};\ x = 1$	$1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n$
$\frac{1}{2+x};\ a = 0$	$\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots + \frac{(-1)^n x^n}{2^{n+1}}$
$\frac{1}{3-2x};\ x = 1$	$1 + 2(x-1) + 4(x-1)^2 + 8(x-1)^3 + \cdots + 2^n(x-1)^n$
$\sqrt{100-2x};\ a = 0$	$10 - \frac{x}{10} - \frac{x^2}{2000} - \frac{x^3}{200000} - \cdots - \frac{(2n-3)!!x^n}{10^{2n-1}n!}$

Definition (Taylor series)

Let $f(x)$ be an infinitely differentiable function. The **Taylor series** of $f(x)$ at $x = a$ is the infinite power series

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots .$$

Example

The following table shows the Taylor series for $f(x)$ at $x = a$.

$f(x)$	Taylor series
$e^x;\ a=0$	$1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$
$\cos x;\ a=0$	$1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots$
$\sin x;\ a=\pi$	$-(x-\pi)+\dfrac{(x-\pi)^3}{3!}-\dfrac{(x-\pi)^5}{5!}+\cdots$
$\ln x;\ a=1$	$(x-1)-\dfrac{(x-1)^2}{2}+\dfrac{(x-1)^3}{3}-\dfrac{(x-1)^4}{4}+\cdots$
$\sqrt{1+x};\ a=0$	$1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}+\cdots$
$\dfrac{1}{\sqrt{1+x}};\ a=0$	$1-\dfrac{x}{2}+\dfrac{3x^2}{8}-\dfrac{5x^3}{16}+\dfrac{35x^4}{128}-\dfrac{63x^5}{256}+\cdots$
$(1+x)^\alpha;\ a=0$	$1+\alpha x+\dfrac{\alpha(\alpha-1)x^2}{2!}+\dfrac{\alpha(\alpha-1)(\alpha-2)x^3}{3!}+\cdots$

$$e^x; \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x; \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x; \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\ln(1+x); \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{1}{1-x}; \quad \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^\alpha; \quad \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \cdots$$

$$\tan^{-1} x; \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\sin^{-1} x; \quad \sum_{k=0}^{\infty} \frac{(2k)! x^{2k+1}}{4^k (k!)^2 (2k+1)} = x + \left(\frac{1}{2}\right)\frac{x^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)\frac{x^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)\frac{x^7}{7} + \cdot$$

Theorem

Suppose $T(x)$ is the Taylor series of $f(x)$ at $x = 0$. Then for any positive integer k, the Taylor series for $f(x^k)$ at $x = 0$ is $T(x^k)$.

Example

$f(x)$	Taylor series at $x = 0$
$\dfrac{1}{1+x^2}$	$1 - x^2 + x^4 - x^6 + \cdots$
$\dfrac{1}{\sqrt{1-x^2}}$	$1 + \dfrac{x^2}{2} + \dfrac{3x^4}{8} + \dfrac{5x^6}{16} + \dfrac{35x^8}{128} + \cdots$
$\dfrac{\sin x^2}{x^2}$	$1 - \dfrac{x^4}{3!} + \dfrac{x^8}{5!} - \dfrac{x^{12}}{7!} + \cdots$

Theorem

Suppose the Taylor series for $f(x)$ at $x = 0$ is

$$T(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Then the Taylor series for $f'(x)$ is

$$T'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

Example

Find the Taylor series of the following functions.

1. $\dfrac{1}{(1+x)^2}$
2. $\tan^{-1} x$

Solution

1. *Let* $F(x) = -\dfrac{1}{1+x}$ *so that* $F'(x) = \dfrac{1}{(1+x)^2}$*. The Taylor series for* $F(x)$ *at* $x = 0$ *is*

$$T(x) = -1 + x - x^2 + x^3 - x^4 + \cdots .$$

Therefore the Taylor series for $F'(x) = \dfrac{1}{(1+x)^2}$ *is*

$$T'(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots .$$

Solution

2. *Suppose the Taylor series for* $f(x) = \tan^{-1} x$ *at* $x = 0$ *is*

$$T(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \cdots .$$

Now comparing $T'(x)$ *with the Taylor series for* $f'(x) = \dfrac{1}{1+x^2}$ *which takes the form*

$$1 - x^2 + x^4 - x^6 + \cdots ,$$

we obtain the values of $a_1, a_2, a_3, \ldots$ *and get*

$$T(x) = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Since $a_0 = f(0) = 0$, *we have*

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

Theorem

Suppose the Taylor series for $f(x)$ *and* $g(x)$ *at* $x = 0$ *are*

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \\ T(x) &= \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots, \end{aligned}$$

respectively. Then the Taylor series for $f(x)g(x)$ *at* $x = 0$ *is*

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \\ = \; & a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots \end{aligned}$$

Proof.

The coefficient of x^n of the Taylor series of $f(x)g(x)$ at $x = 0$ is

$$\begin{aligned}
\frac{(fg)^{(n)}(0)}{n!} &= \sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \quad \text{(Leibniz's formula)} \\
&= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot \frac{g^{(n-k)}(0)}{(n-k)!} \\
&= \sum_{k=0}^{n} a_k b_{n-k}
\end{aligned}$$

□

Example

1. The Taylor series for $e^{4x}\ln(1+x)$ is

$$\begin{aligned} & \left(1+4x+\frac{16x^2}{2!}+\frac{64x^3}{3!}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right) \\ = \quad & x+\left(-\frac{1}{2}+4\right)x^2+\left(\frac{1}{3}+4\cdot\left(-\frac{1}{2}\right)+8\right)x^3+\cdots \\ = \quad & x+\frac{7x^2}{2}+\frac{19x^3}{3}+\cdots \end{aligned}$$

2. The Taylor series for $\dfrac{\tan^{-1}x}{\sqrt{1-x^2}}$ is

$$\begin{aligned} & \left(x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right)\left(1+\frac{x^2}{2}+\frac{3x^4}{8}+\cdots\right) \\ = \quad & x+\left(\frac{1}{2}-\frac{1}{3}\right)x^3+\left(\frac{3}{4}-\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{5}\right)x^5+\cdots \\ = \quad & x+\frac{x^3}{6}+\frac{49x^5}{120}+\cdots \end{aligned}$$

Theorem

Suppose $f(x)$ and $g(x)$ are infinitely differentiable functions and the Taylor series of $f(x)$ and $g(x)$ at $x = 0$ are

$$a_k x^k + a_{k+1}x^{k+1} + a_{k+2}x^{k+2} + \cdots$$

and

$$b_k x^k + b_{k+1}x^{k+1} + b_{k+2}x^{k+2} + \cdots$$

where $b_k \neq 0$. Then

$$\begin{aligned} \lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \frac{a_k + a_{k+1}x + a_{k+2}x^2 + \cdots}{b_k + b_{k+1}x + b_{k+2}x^2 + \cdots} \\ &= \frac{a_k}{b_k} \end{aligned}$$

Proof.

The assumptions on $f(x)$ and $g(x)$ imply that

$$f(0) = f'(0) = f''(0) = \cdots = f^{(k-1)}(0) = 0; \;\; f^{(k)}(0) = a_k$$
$$g(0) = g'(0) = g''(0) = \cdots = g^{(k-1)}(0) = 0; \;\; g^{(k)}(0) = b_k$$

Therefore, by L'Hopital's rule, we have

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \cdots = \lim_{x\to 0} \frac{f^{(k)}(x)}{g^{(k)}(x)} = \frac{a_k}{b_k}.$$

□

Example

1.
$$\begin{aligned}
&\lim_{x\to 0}\frac{\ln(1+x)-x\sqrt{1-x}}{x-\sin x}\\
=\ &\lim_{x\to 0}\frac{(x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots)-x(1-\frac{x}{2}-\frac{x^2}{8}+\cdots)}{x-(x-\frac{x^3}{6}+\cdots)}\\
=\ &\lim_{x\to 0}\frac{\frac{11x^3}{24}+\cdots}{\frac{x^3}{6}+\cdots}\\
=\ &\frac{11}{4}
\end{aligned}$$

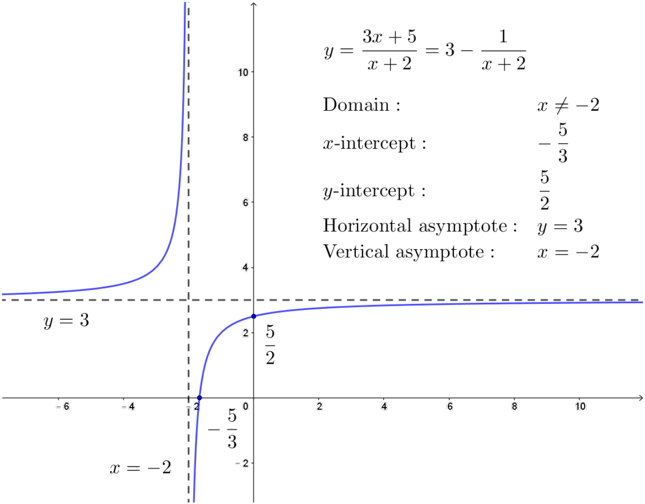
2.
$$\begin{aligned}
&\lim_{x\to 0}\left(\frac{e^x}{x}-\frac{1}{\tan x}\right)=\lim_{x\to 0}\frac{e^x\sin x-x\cos x}{x\sin x}\\
=\ &\lim_{x\to 0}\frac{(1+x+\frac{x^2}{2}+\cdots)(x-\frac{x^3}{6}+\cdots)-x(1-\frac{x^2}{2}+\cdots)}{x(x-\frac{x^3}{6}+\cdots)}\\
=\ &\lim_{x\to 0}\frac{(x+x^2+\frac{x^3}{3}+\cdots)-(x-\frac{x^3}{2}+\cdots)}{x^2-\frac{x^4}{6}+\cdots}\\
=\ &\lim_{x\to 0}\frac{x^2+\frac{5x^3}{6}+\cdots}{x^2-\frac{x^4}{6}+\cdots}\\
=\ &1
\end{aligned}$$

Curve sketching

To sketch the graph of $y = f(x)$, one first finds

- Domain: The values of x where $f(x)$ is defined.
- x-intercepts: The values of x such that $f(x) = 0$.
- y-intercept: $f(0)$
- Horizontal asymptotes:
  If $\lim_{x\to-\infty/+\infty} f(x) = b$, then $y = b$ is a horizontal asymptote.
- Vertical asymptotes:
  If $\lim_{x\to a^-/a^+} f(x) = -\infty/+\infty$, then $x = a$ is a vertical asymptote.

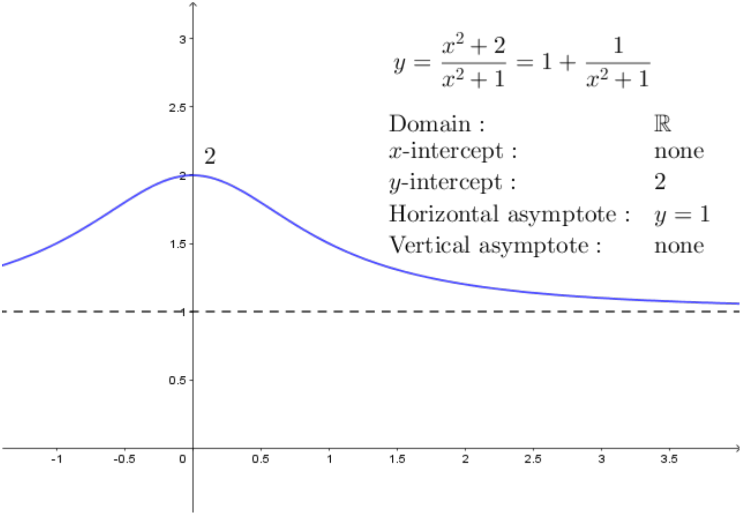
Example 1: $f(x) = \dfrac{3x+5}{x+2}$



$$y = \frac{3x+5}{x+2} = 3 - \frac{1}{x+2}$$

Domain :	$x \neq -2$
x-intercept :	$-\frac{5}{3}$
y-intercept :	$\frac{5}{2}$
Horizontal asymptote :	$y = 3$
Vertical asymptote :	$x = -2$

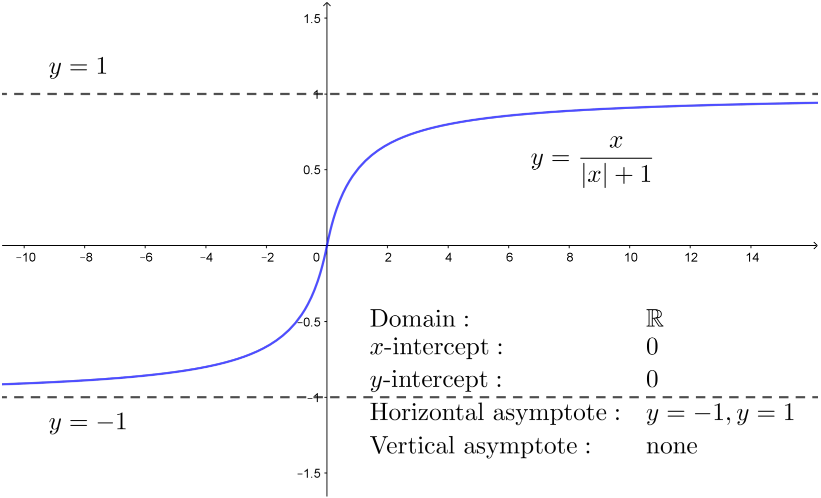
Example 2: $f(x) = \dfrac{x^2+2}{x^2+1}$



$$y = \frac{x^2+2}{x^2+1} = 1 + \frac{1}{x^2+1}$$

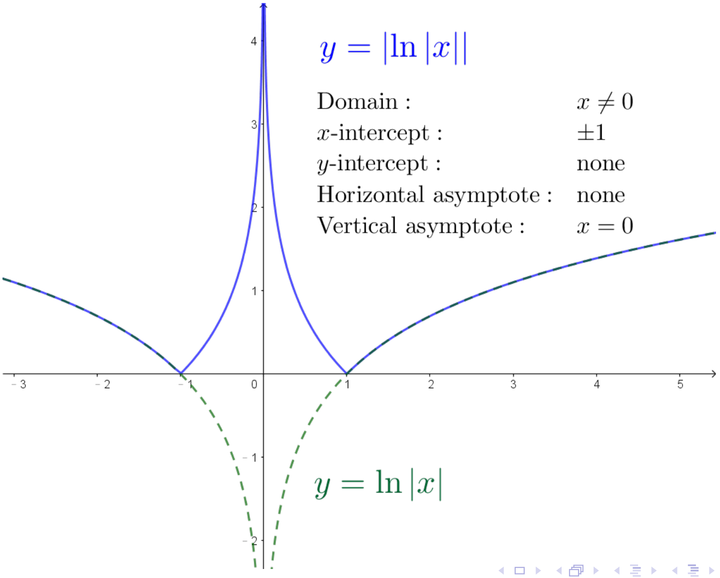
Domain :	$\mathbb{R}$
x-intercept :	none
y-intercept :	2
Horizontal asymptote :	$y = 1$
Vertical asymptote :	none

Example 3: $f(x) = \dfrac{x}{|x|+1}$



Domain :	$\mathbb{R}$
x-intercept :	0
y-intercept :	0
Horizontal asymptote :	$y = -1, y = 1$
Vertical asymptote :	none

Example 4: $f(x) = |\ln|x||$

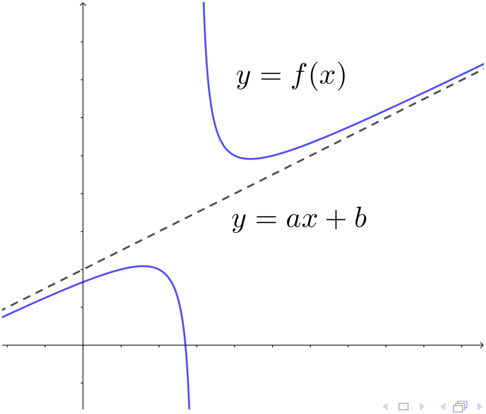


Definition (Oblique asymptote)

If

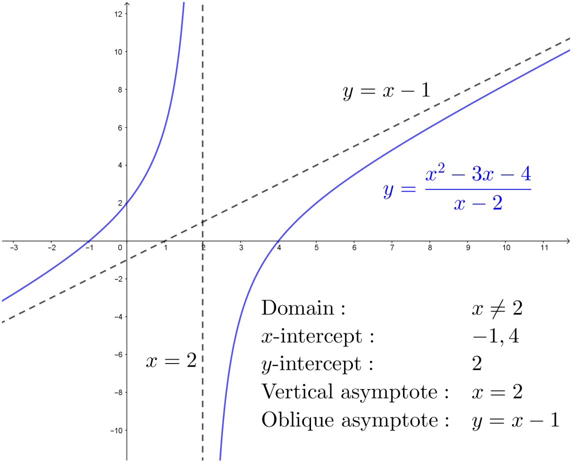
$$\lim_{x\to-\infty/+\infty}(f(x)-(ax+b))=0,$$

we say that $y=ax+b$ is an oblique asymptote of $y=f(x)$.



Example 5: $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$.

Note that $\dfrac{x^2 - 3x - 4}{x - 2} = \dfrac{x^2 - 2x - (x - 2) - 6}{x - 2} = x - 1 - \dfrac{6}{x - 2}$.

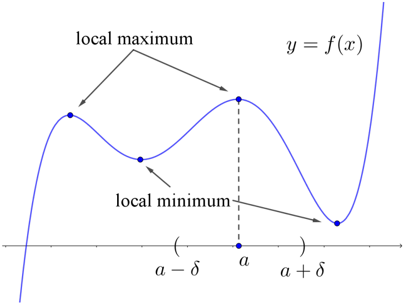


Domain :	$x \neq 2$
x-intercept :	$-1, 4$
y-intercept :	2
Vertical asymptote :	$x = 2$
Oblique asymptote :	$y = x - 1$

Definition

Let $f(x)$ be a continuous function. We say that $f(x)$ has a

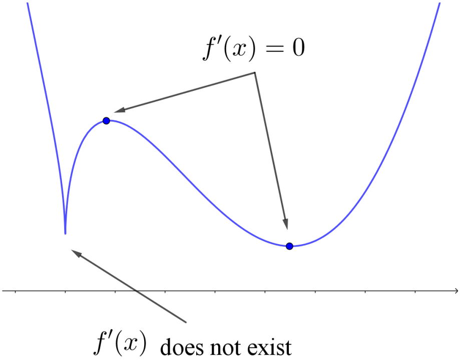
1. **local maximum** at $x = a$ if there exists $\delta > 0$ such that $f(x) \leq f(a)$ for any $x \in (a - \delta, a + \delta)$.
2. **local minimum** at $x = a$ if there exists $\delta > 0$ such that $f(x) \geq f(a)$ for any $x \in (a - \delta, a + \delta)$.



Theorem

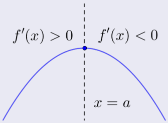
Let $f(x)$ be a continuous function. Suppose $f(x)$ has local maximum or local minimum at $x = a$. Then either

1. $f'(a) = 0$, *or*
2. $f'(x)$ *does not exist at* $x = a$.



Theorem (First derivative test)

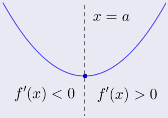
Let $f(x)$ be a continuous function and $f'(a) = 0$ or $f'(a)$ does not exist. Suppose there is $\delta > 0$ such that



1.

	$a - \delta < x < a$	$a < x < a + \delta$
$f'(x)$	$+$	$-$

Then $f(x)$ has a local maximum at $x = a$.



2.

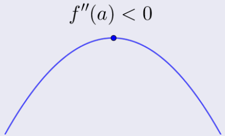
	$a - \delta < x < a$	$a < x < a + \delta$
$f'(x)$	$-$	$+$

Then $f(x)$ has a local minimum at $x = a$.

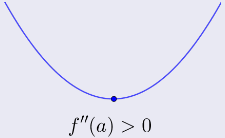
Theorem (Second derivative test)

Let $f(x)$ *be a differentiable function and* $f'(a) = 0$*.*

1. *If* $f''(a) < 0$*, then* $f(x)$ *has a local maximum at* $x = a$*.*



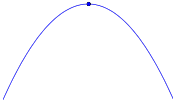
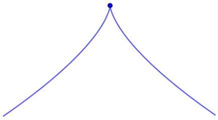
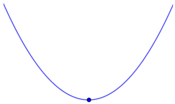
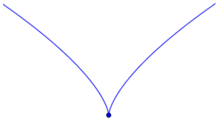
2. *If* $f''(a) > 0$*, then* $f(x)$ *has a local minimum at* $x = a$*.*



Definition (Turning point)

We say that $f(x)$ has a **turning point** at $x = a$ if $f'(x)$ changes sign at $x = a$.

If $f(x)$ has a turning point at $x = a$, then either $f'(a) = 0$ or $f'(x)$ does not exist.

Turning point	$f'(a) = 0$	$f'(a)$ does not exist
Relative maximum		
Relative minimum		

Example 6: $f(x) = \dfrac{x-3}{x^2+4x-5}$

$$f(x) = \frac{x-3}{(x-1)(x+5)},\ x \neq -5, 1$$

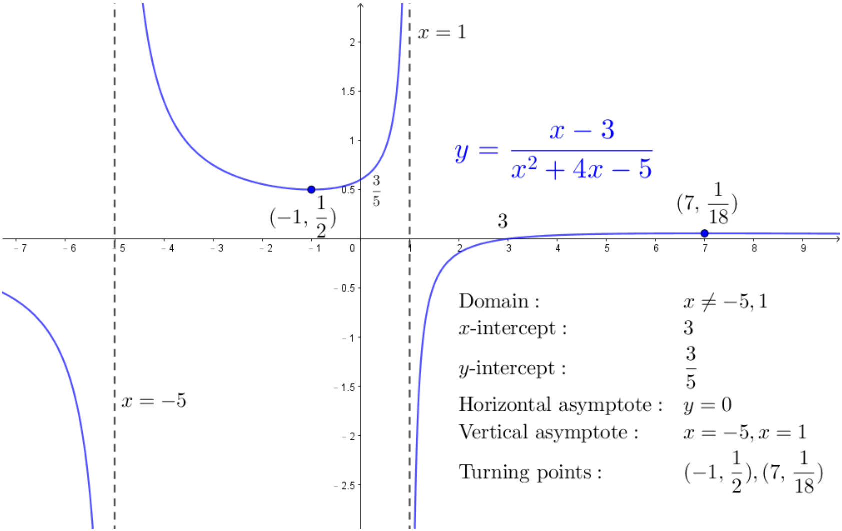
$$f'(x) = \frac{(x^2+4x-5)(1)-(x-3)(2x+4)}{(x-1)^2(x+5)^2} = -\frac{(x+1)(x-7)}{(x-1)^2(x+5)^2}$$

Thus $f'(x) = 0$ when $x = -1, 7$.

	$x < -5$	$-5 < x < -1$	$-1 < x < 1$	$1 < x < 7$	$x > 7$
$f'(x)$	$-$	$-$	$+$	$+$	$-$

$(-1, \frac{1}{2})$ is a minimum point and $(7, \frac{1}{18})$ is a maximum point.

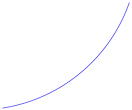
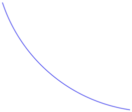
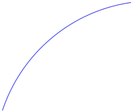
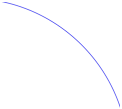
Example: $f(x) = \dfrac{x-3}{x^2+4x-5}$.



Definition (Concavity)

We say that $f(x)$ is

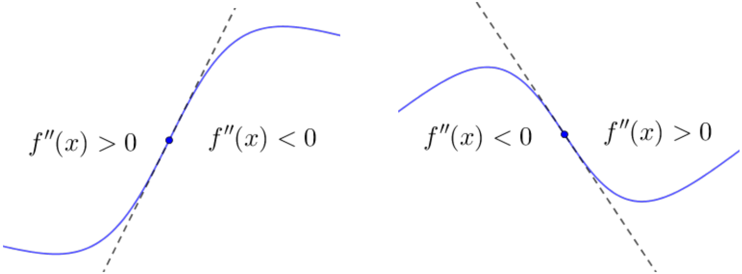
1. **Concave upward** on (a, b) if $f''(x) > 0$ on (a, b).
2. **Concave downward** on (a, b) if $f''(x) < 0$ on (a, b).

	$f'(x) > 0$	$f'(x) < 0$
Concave upward ($f''(x) > 0$)		
Concave downward ($f''(x) < 0$)		

Definition (Inflection point)

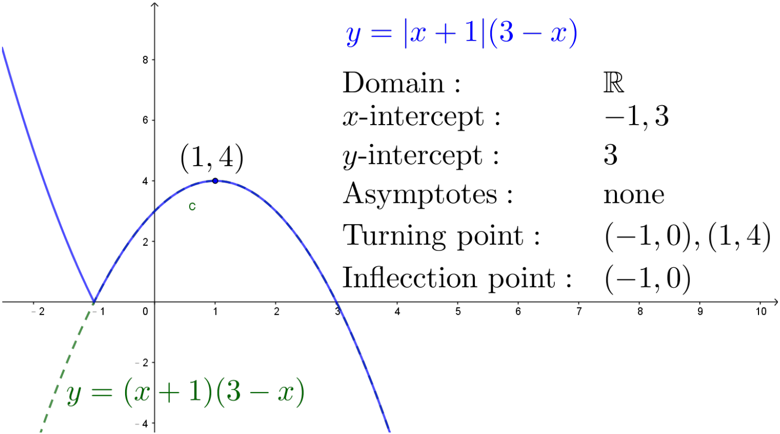
We say that $f(x)$ has an **inflection point** at $x = a$ if $f''(x)$ changes sign at $x = a$.

If $f(x)$ has an inflection point at $x = a$, then ether $f''(a) = 0$ or $f''(a)$ does not exist.



Example 7: $f(x) = |x+1|(3-x)$

$$f(x) = |x+1|(3-x) = \begin{cases} (x+1)(x-3) & \text{if } x < -1 \\ -(x+1)(x-3) & \text{if } x \geq -1 \end{cases}$$



$y = |x+1|(3-x)$

Domain :	$\mathbb{R}$
x-intercept :	$-1, 3$
y-intercept :	3
Asymptotes :	none
Turning point :	$(-1,0), (1,4)$
Inflecction point :	$(-1,0)$

Example 8: $f(x) = x + \frac{1}{|x|}$

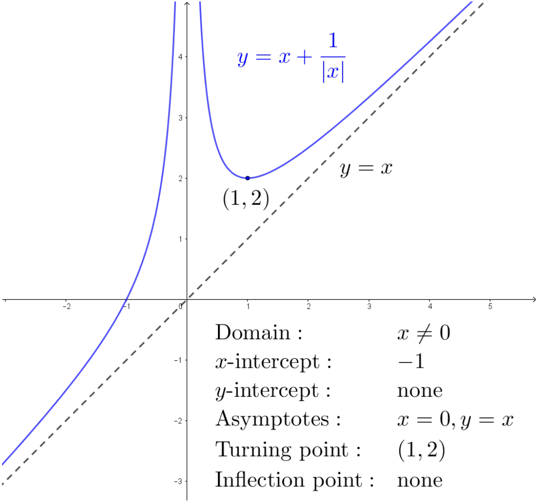
Since $\lim_{x\to\pm\infty}(f(x) - x) = \lim_{x\to\pm\infty}\frac{1}{|x|} = 0$,
$y = f(x)$ has an oblique asymptote $y = x$.
When $x < 0$, $f(x) = x - \frac{1}{x}$.
$f'(x) = 1 + \frac{1}{x^2}$
$f''(x) = -\frac{2}{x^3}$
When $x > 0$, $f(x) = x + \frac{1}{x}$.
$f'(x) = 1 - \frac{1}{x^2}$
$f''(x) = \frac{2}{x^3}$

	$x < 0$	$0 < x < 1$	$x > 1$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$+$	$+$	$+$

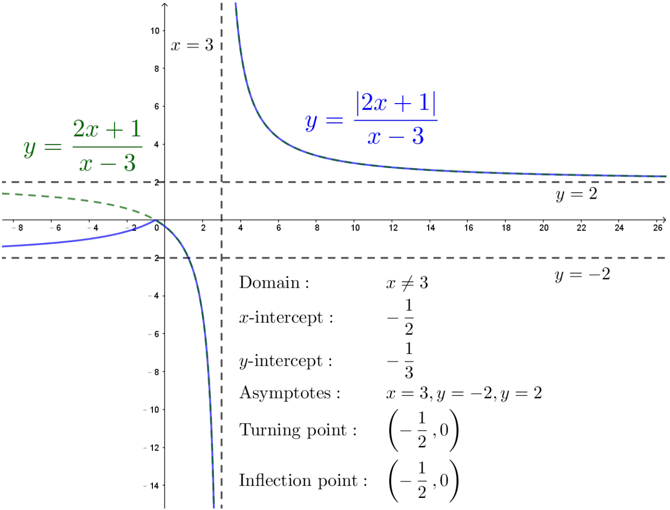
$f(x)$ has a minimum point at $x = 1$.

$f(x)$ has no inflection point.

Example 8: $f(x) = x + \dfrac{1}{|x|}$



Example 9: $f(x) = \dfrac{|2x+1|}{x-3}$



Domain :	$x \neq 3$
x-intercept :	$-\dfrac{1}{2}$
y-intercept :	$-\dfrac{1}{3}$
Asymptotes :	$x = 3, y = -2, y = 2$
Turning point :	$\left(-\dfrac{1}{2}, 0\right)$
Inflection point :	$\left(-\dfrac{1}{2}, 0\right)$

Example 10: $f(x) = 2 - (x-8)^{\frac{1}{3}}$

$f'(x) = -\dfrac{1}{3(x-8)^{\frac{2}{3}}}$

$f''(x) = \dfrac{2}{9(x-8)^{\frac{5}{3}}}$

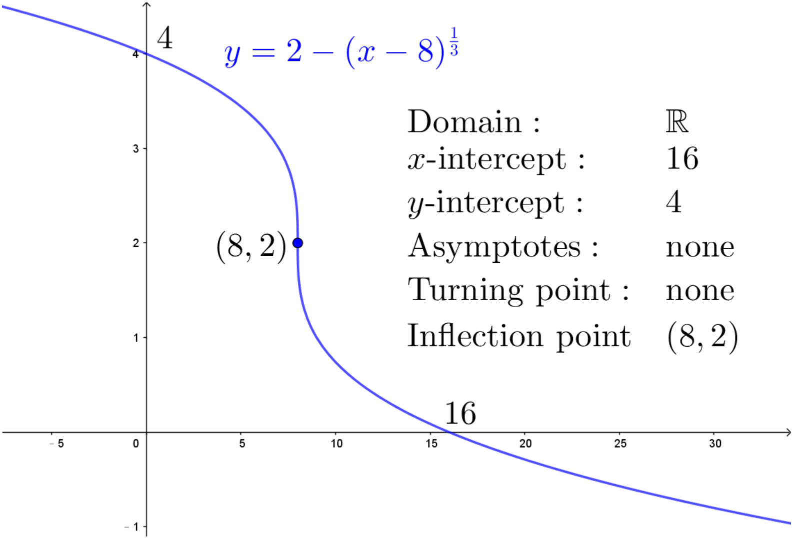
$f'(x)$, $f''(x)$ do not exist at $x = 8$.

	$x < 8$	$x > 8$
$f'(x)$	$-$	$-$
$f''(x)$	$-$	$+$

$f(x)$ has no turning point.

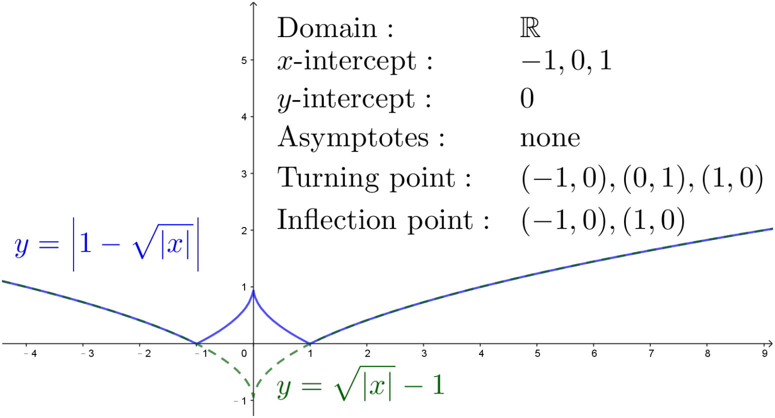
$f(x)$ has an inflection point at $x = 8$.

Example 10: $f(x) = 2 - (x-8)^{\frac{1}{3}}$



Domain :	$\mathbb{R}$
x-intercept :	16
y-intercept :	4
Asymptotes :	none
Turning point :	none
Inflection point	$(8, 2)$

Example 11: $f(x) = \left|1 - \sqrt{|x|}\right|$



Example 12: $f(x) = \dfrac{x^2 + x - 2}{x^2}$

Domain: $x \neq 0$

$f(x) = \dfrac{x^2 + x - 2}{x^2} = 1 + \dfrac{x - 2}{x^2}$

$f(x)$ has a horizontal asymptote $y = 1$.

$f'(x) = \dfrac{x^2 - 2x(x-2)}{x^4} = \dfrac{x - 2(x-2)}{x^3} = -\dfrac{x-4}{x^3}$

$f'(x) = 0$ when $x = 4$

$f''(x) = -\dfrac{x^3 - 3x^2(x-4)}{x^6} = -\dfrac{x - 3(x-4)}{x^6} = \dfrac{2(x-6)}{x^4}$

$f''(x) = 0$ when $x = 6$.

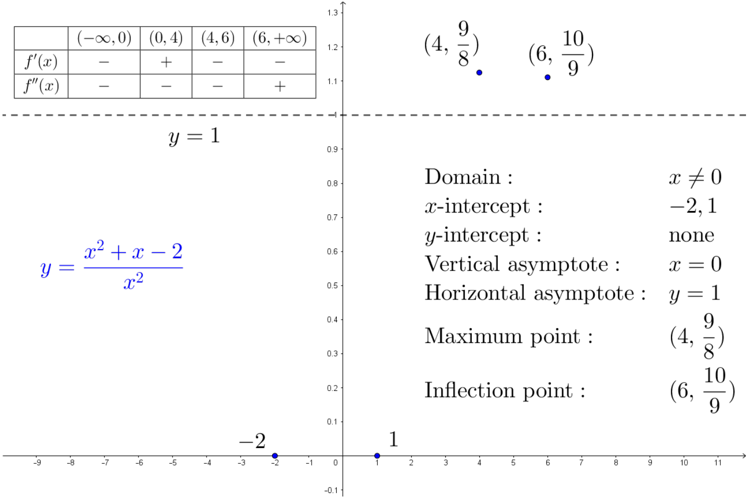
	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$

$(4, \frac{9}{8})$ is maximum point.

$(6, \frac{10}{9})$ is an inflection point.

Example 12: $f(x) = \dfrac{x^2+x-2}{x^2}$

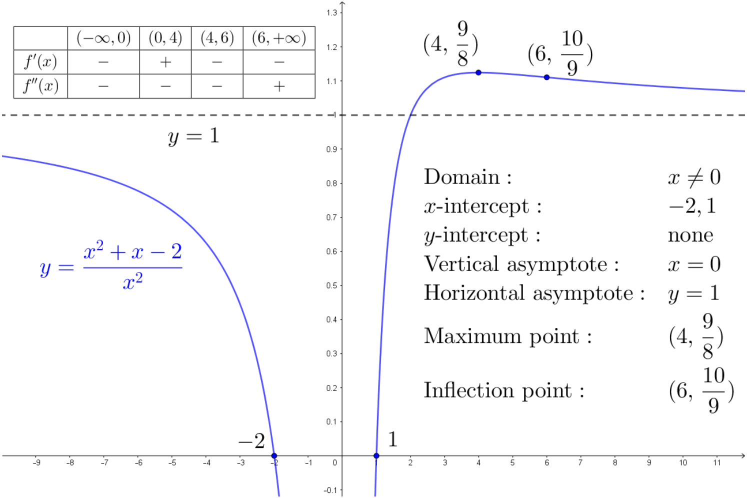
	$(-\infty, 0)$	$(0,4)$	$(4,6)$	$(6,+\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$



Domain :	$x \neq 0$
x-intercept :	$-2, 1$
y-intercept :	none
Vertical asymptote :	$x = 0$
Horizontal asymptote :	$y = 1$
Maximum point :	$(4, \frac{9}{8})$
Inflection point :	$(6, \frac{10}{9})$

Example 12: $f(x) = \dfrac{x^2 + x - 2}{x^2}$

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$



Example 13: $f(x) = \dfrac{x^3}{(x-2)^2}$

$f(x) = x + 4 + \dfrac{12x - 16}{(x-2)^2}$, $x \neq 2$

$f(x)$ has an oblique asymptote $y = x + 4$

$$f'(x) = \frac{3x^2(x-2)^2 - 2(x-2)x^3}{(x-2)^4} = \frac{3x^2(x-2) - 2x^3}{(x-2)^3} = \frac{x^3 - 6x^2}{(x-2)^3}$$

$f'(x) = 0$ when $x = 0, 6$

$$f''(x) = \frac{(3x^2 - 12x)(x-2)^3 - 3(x-2)^2(x^3 - 6x^2)}{(x-2)^6} = \frac{24x}{(x-2)^4}$$

$f''(x) = 0$ when $x = 0$.

	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$-$	$+$	$+$	$+$

$(6, \frac{27}{2})$ is minimum point.

$(0, 0)$ is an inflection point.

Example 13: $f(x) = \dfrac{x^3}{(x-2)^2}$

Domain : $x \neq 2$

x-intercept : 0

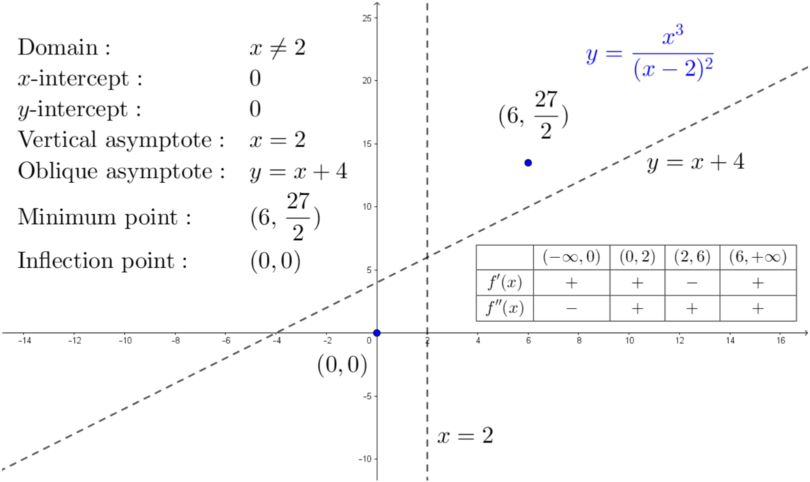
y-intercept : 0

Vertical asymptote : $x = 2$

Oblique asymptote : $y = x + 4$

Minimum point : $(6, \frac{27}{2})$

Inflection point : $(0, 0)$



	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$-$	$+$	$+$	$+$

Example 13: $f(x) = \dfrac{x^3}{(x-2)^2}$

Domain : $x \neq 2$

x-intercept : 0

y-intercept : 0

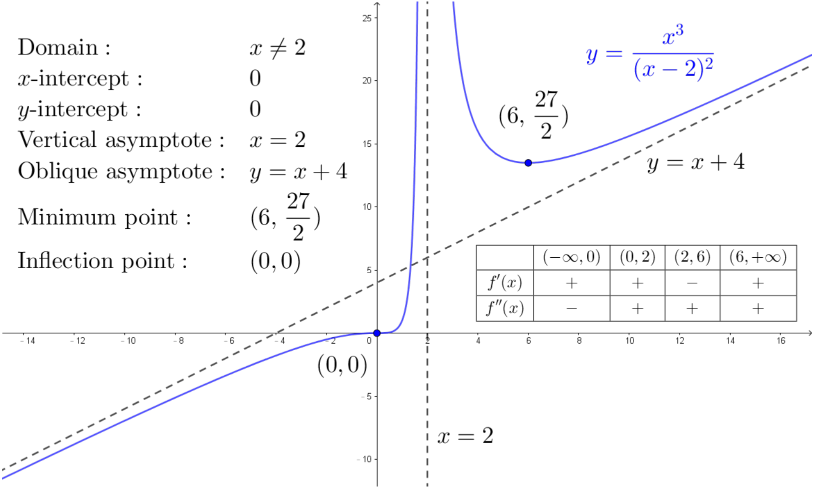
Vertical asymptote : $x = 2$

Oblique asymptote : $y = x + 4$

Minimum point : $(6, \frac{27}{2})$

Inflection point : $(0, 0)$

	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$-$	$+$	$+$	$+$



Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$
First

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}}{x} = \lim_{x\to\pm\infty} \left(1-\frac{3}{x}\right)^{\frac{2}{3}} = 1$$

and

$$\begin{aligned} \lim_{x\to\pm\infty}(f(x)-x) &= \lim_{x\to\pm\infty} x\left(\left(1-\frac{3}{x}\right)^{\frac{2}{3}}-1\right) \\ &= \lim_{h\to 0}\frac{(1-3h)^{\frac{2}{3}}-1}{h} \\ &= -2 \end{aligned}$$

Thus $y = x - 2$ is an oblique asymptote.

Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-\frac{2}{3}}(x-3)^{\frac{2}{3}} + \frac{2}{3}x^{\frac{1}{3}}(x-3)^{-\frac{1}{3}} \\ &= \frac{x-1}{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}}} \end{aligned}$$

$f'(x) = 0$ when $x = 1$ and $f'(x)$ does not exist when $x = 0, 3$.

$$\begin{aligned} f''(x) &= \frac{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}} - (\frac{2}{3}x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}})(x-1)}{x^{\frac{4}{3}}(x-3)^{\frac{2}{3}}} \\ &= \frac{3x(x-3) - (2(x-3)+x)(x-1)}{3x^{\frac{5}{3}}(x-3)^{\frac{4}{3}}} \\ &= -\frac{2}{x^{\frac{5}{3}}(x-3)^{\frac{4}{3}}} \end{aligned}$$

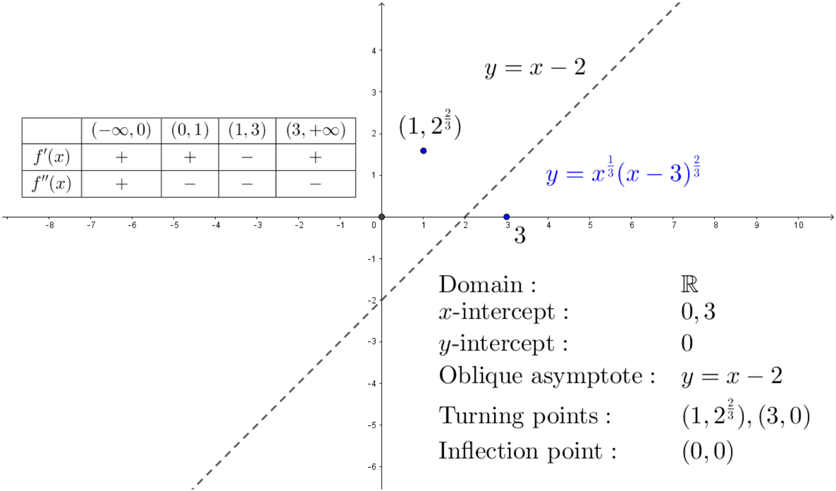
$f''(x)$ does not exist when $x = 0, 3$.

	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$-$	$-$

$(1, 2^{\frac{2}{3}})$ is a maximum point.
$(3, 0)$ is a minimum point.

$(0, 0)$ is an inflection point.

Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

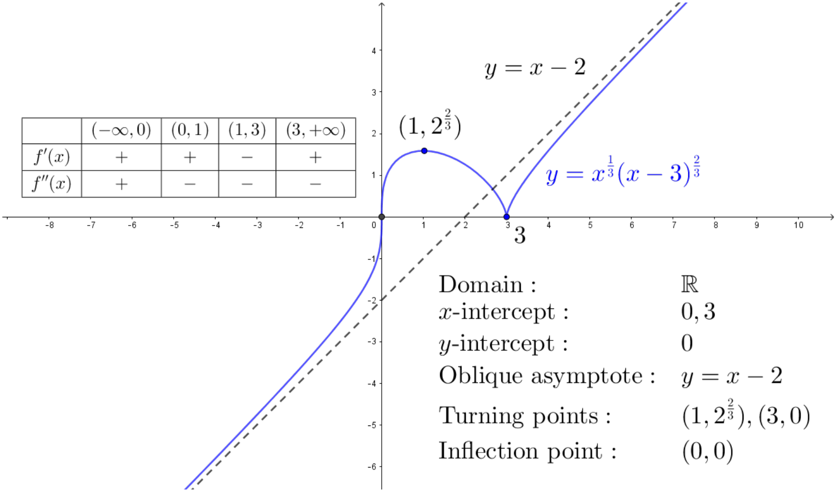


	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, +\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$-$	$-$

Domain :	$\mathbb{R}$
x-intercept :	$0, 3$
y-intercept :	0
Oblique asymptote :	$y = x - 2$
Turning points :	$(1, 2^{\frac{2}{3}}), (3, 0)$
Inflection point :	$(0, 0)$

Example 14: $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

	$(-\infty, 0)$	$(0,1)$	$(1,3)$	$(3,+\infty)$
$f'(x)$	$+$	$+$	$-$	$+$
$f''(x)$	$+$	$-$	$-$	$-$



Domain :	$\mathbb{R}$
x-intercept :	$0, 3$
y-intercept :	0
Oblique asymptote :	$y = x - 2$
Turning points :	$(1, 2^{\frac{2}{3}}), (3, 0)$
Inflection point :	$(0, 0)$

Indefinite integral and substitution

Definition

Let $f(x)$ be a continuous function. A **primitive function**, or an **anti-derivative**, of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

The collection of all anti-derivatives of $f(x)$ is called the **indefinite integral** of $f(x)$ and is denoted by

$$\int f(x)dx.$$

The function $f(x)$ is called the **integrand** of the integral.

Note: Anti-derivative of a function is not unique. If $F(x)$ is an anti-derivative of f, then $F(x)+C$ is an anti-derivative of $f(x)$ for any constant C. Moreover, any anti-derivative of $f(x)$ is of the form $F(x)+C$ and we write

$$\int f(x)dx = F(x) + C$$

where C is arbitrary constant called the **integration constant**. Note that $\int f(x)dx$ is not a single function but a collection of functions.

Theorem

Let $f(x)$ and $g(x)$ be continuous functions and k be a constant.

1. $$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$
2. $$\int kf(x)dx = k\int f(x)dx$$

Theorem (formulas for indefinite integrals)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$$

$$\int e^x dx = e^x + C; \qquad \int \frac{1}{x}dx = \ln|x| + C$$

$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C; \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C; \qquad \int \csc x \cot x dx = -\csc x + C$$

Example

1. $$\int (x^3 - x + 5)dx = \frac{x^4}{4} - \frac{x^2}{2} + 5x + C$$

2. $$\int \frac{(x+1)^2}{x}dx = \int \frac{x^2 + 2x + 1}{x}dx = \int \left(x + 2 + \frac{1}{x}\right) dx = \frac{x^2}{2} + 2x + \ln|x| + C$$

3. $$\int \frac{3x^2 + \sqrt{x} - 1}{\sqrt{x}}dx = \int \left(3x^{3/2} + 1 - x^{-1/2}\right) dx = \frac{6}{5}x^{\frac{5}{2}} + x - 2x^{\frac{1}{2}} + C$$

4. $$\int \left(\frac{3\sin x}{\cos^2 x} - 2e^x\right) dx = \int (3\sec x \tan x - 2e^x)\, dx = 3\sec x - 2e^x + C$$

Example

Suppose we want to compute

$$\int x\sqrt{x^2+4}\,dx$$

First we let

$$u = x^2 + 4.$$

We may formally write

$$du = \frac{du}{dx}\,dx = \left[\frac{d}{dx}(x^2+4)\right]dx = 2xdx$$

Here du is called the differential of u defined as $\dfrac{du}{dx}\,dx$. Thus the integral is

$$\begin{aligned}\int x\sqrt{x^2+4}\,dx &= \frac{1}{2}\int \sqrt{x^2+4}(2xdx) = \frac{1}{2}\int \sqrt{u}\,du \\ &= \frac{u^{\frac{3}{2}}}{3} + C = \frac{(x^2+4)^{\frac{3}{2}}}{3} + C\end{aligned}$$

Example

$$\begin{aligned}
\int x\sqrt{x^2+4}\,dx &= \int \sqrt{x^2+4}\,d\left(\frac{x^2}{2}\right)\\
&= \frac{1}{2}\int \sqrt{x^2+4}\,dx^2\\
&= \frac{1}{2}\int \sqrt{x^2+4}\,d(x^2+4)\\
&= \frac{(x^2+4)^{\frac{3}{2}}}{3}+C
\end{aligned}$$

Theorem

Let $f(x)$ be a continuous function defined on $[a,b]$. Suppose there exists a differentiable function $u=\varphi(x)$ and continuous function $g(u)$ such that $f(x)=g(\varphi(x))\varphi'(x)$ for any $x\in(a,b)$. Then

$$\begin{aligned}\int f(x)dx &= \int g(\varphi(x))\varphi'(x)dx \\ &= \int g(u)du\end{aligned}$$

Example

$$\int x^2 e^{x^3+1} dx$$

Let $u = x^3 + 1$,

then $du = 3x^2 dx$

$$= \frac{1}{3} \int e^u du$$

$$= \frac{e^u}{3} + C$$

$$= \frac{e^{x^3+1}}{3} + C$$

$$\int x^2 e^{x^3+1} dx$$

$$= \int e^{x^3+1} d\left(\frac{x^3}{3}\right)$$

$$= \frac{1}{3} \int e^{x^3+1} dx^3$$

$$= \frac{1}{3} \int e^{x^3+1} d(x^3+1)$$

$$= \frac{e^{x^3+1}}{3} + C$$

Example

$$\int \cos^4 x \sin x dx$$

Let $u = \cos x$,

then $du = -\sin x dx$

$$= -\int u^4 du$$

$$= -\frac{u^5}{5} + C$$

$$= -\frac{\cos^5 x}{5} + C$$

$$\int \cos^4 x \sin x dx$$

$$= \int \cos^4 x d(-\cos x)$$

$$= -\int \cos^4 x d \cos x$$

$$= -\frac{\cos^5 x}{5} + C$$

Example

$$\int \frac{dx}{x \ln x}$$

Let $u = \ln x$,

then $du = \dfrac{dx}{x}$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

$$\int \frac{dx}{x \ln x}$$

$$= \int \frac{d \ln x}{\ln x}$$

$$= \ln|\ln x| + C$$

Example

$$\int \frac{dx}{e^x+1}$$

Let $u = 1 + e^{-x}$,

then $du = -e^{-x}dx$

$$\begin{aligned}
&= \int \frac{e^{-x}dx}{1+e^{-x}} \\
&= -\int \frac{du}{u} \\
&= -\ln u + C \\
&= -\ln(1+e^{-x}) + C \\
&= x - \ln(1+e^x) + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{dx}{e^x+1} \\
&= \int \left(1 - \frac{e^x}{1+e^x}\right) dx \\
&= x - \int \frac{de^x}{1+e^x} \\
&= x - \ln(1+e^x) + C
\end{aligned}$$

Example

$$\int \frac{dx}{1+\sqrt{x}}$$

Let $u = 1+\sqrt{x}$,

then $du = \dfrac{dx}{2\sqrt{x}}$

$$\begin{aligned}
&= 2\int \frac{(u-1)du}{u} \\
&= 2\int \left(1-\frac{1}{u}\right) du \\
&= 2u - 2\ln u + C' \\
&= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{dx}{1+\sqrt{x}} \\
&= \int \frac{\sqrt{x}\,dx}{\sqrt{x}(1+\sqrt{x})} \\
&= 2\int \frac{\sqrt{x}\,d\sqrt{x}}{1+\sqrt{x}} \\
&= 2\int \left(1-\frac{1}{1+\sqrt{x}}\right) d\sqrt{x} \\
&= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C
\end{aligned}$$

Definite integral

Definition

Let $f(x)$ be a function on $[a, b]$. A **Partition** of $[a, b]$ is a set of finite points

$$P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$$

and we define

$$\begin{aligned} \Delta x_k &= x_k - x_{k-1}, \text{ for } k = 1, 2, \ldots, n \\ \|P\| &= \max_{1 \le k \le n} \{\Delta x_k\} \end{aligned}$$

Definition

Let $f(x)$ be a function on $[a,b]$. The **lower** and **upper Riemann sums** with respect to partition P are

$$\mathcal{L}(f,P)=\sum_{k=1}^{n} m_k \Delta x_k, \text{ and } \mathcal{U}(f,P)=\sum_{k=1}^{n} M_k \Delta x_k$$

where

$$m_k=\inf\{f(x): x_{k-1}\le x\le x_k\}, \text{ and } M_k=\sup\{f(x): x_{k-1}\le x\le x_k\}$$

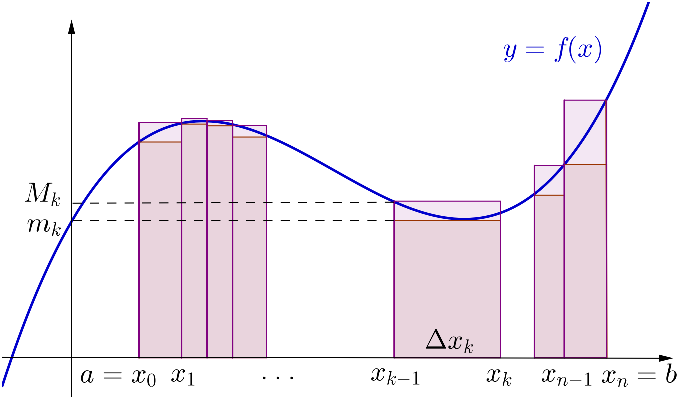


Figure: Upper and lower Riemann sum

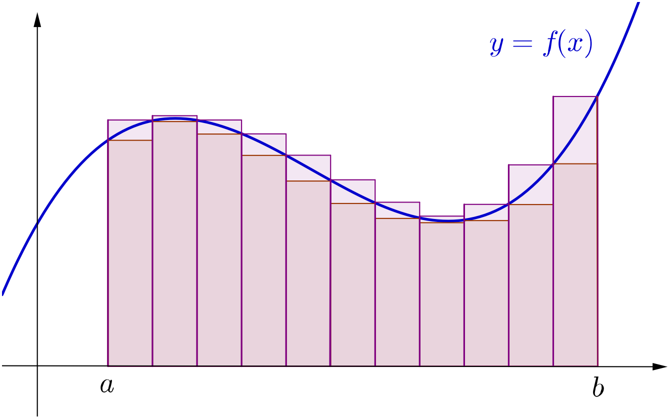


Figure: Upper and lower Riemann sum

Definition (Riemann integral)

Let $[a,b]$ be a closed and bounded interval and $f:[a,b]\to\mathbb{R}$ be a real valued function defined on $[a,b]$. We say that $f(x)$ is **Riemann integrable** on $[a,b]$ if the limits of $\mathcal{L}(f,P)$ and $\mathcal{U}(f,P)$ *exist* as $\|P\|$ tends to 0 and are *equal*. In this case, we define the **Riemann integral** of $f(x)$ over $[a,b]$ by

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0}\mathcal{L}(f,P) = \lim_{\|P\|\to 0}\mathcal{U}(f,P).$$

Note: We say that $\lim_{\|P\|\to 0}\mathcal{L}(f,P) = L$ if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|P\| < \delta$, then $|\mathcal{L}(f,P) - L| < \varepsilon$.

Theorem

Let $f(x)$ *and* $g(x)$ *be integrable functions on* $[a,b]$, $a < c < b$ *and* k *be constants.*

1. $\displaystyle\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$
3. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$

Theorem

Suppose $f(x)$ is a continuous function on $[a, b]$. Then $f(x)$ is Riemann integrable on $[a, b]$ and we have

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+\frac{k}{n}(b-a)\right)\left(\frac{b-a}{n}\right).\end{aligned}$$

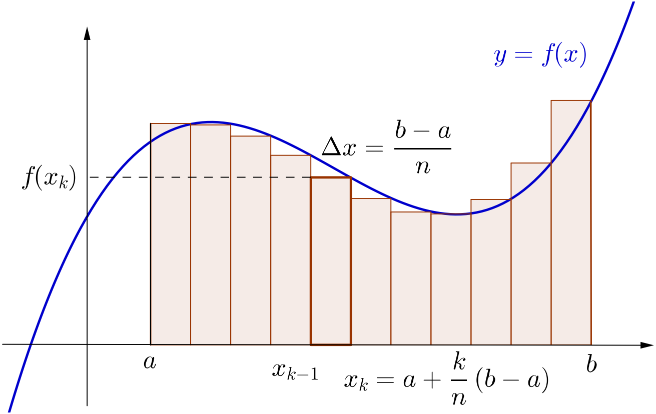


Figure: Formula for Riemann integral

Example

Use the formula for definite integral of continuous function to evaluate

$$\int_0^1 x^2 dx$$

Solution

$$\begin{aligned}
\int_0^1 x^2 dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(0 + \frac{k}{n}(1-0)\right)^2 \left(\frac{1-0}{n}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&= \frac{1}{3}
\end{aligned}$$

Fundamental theorem of calculus

Theorem (Fundamental theorem of calculus)

First part: *Let $f(x)$ be a function which is continuous on $[a,b]$. Let $F : [a,b] \to \mathbb{R}$ be the function defined by*

$$F(x) = \int_a^x f(t)dt$$

Then $F(x)$ is continuous on $[a,b]$, differentiable on (a,b) and

$$F'(x) = f(x).$$

for any $x \in (a,b)$. Put in another way, we have

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \quad \text{for } x \in (a,b).$$

Theorem (Fundamental theorem of calculus)

Second part: *Let $f(x)$ be a function which is continuous on $[a, b]$. Let $F(x)$ be a primitive function of $f(x)$, in other words, $F(x)$ is a continuous function on $[a, b]$ and $F'(x) = f(x)$ for any $x \in (a, b)$. Then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example

Let $f(x) = \sqrt{1-x^2}$. The graph of $y = f(x)$ is a unit semicircle centered at the origin. Using the formula for area of circular sectors, we calculate

$$F(x) = \int_0^x f(t)dt = \int_0^x \sqrt{1-t^2}dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2}.$$

By fundamental theorem of calculus, we know that $F(x)$ is an anti-derivative of $f(x)$. One may check this by differentiating $F(x)$ and get

$$\begin{aligned} F'(x) &= \frac{1}{2}\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) \\ &= \frac{1}{2}\left(\frac{1-x^2-x^2+1}{\sqrt{1-x^2}}\right) \\ &= \sqrt{1-x^2} \\ &= f(x) \end{aligned}$$

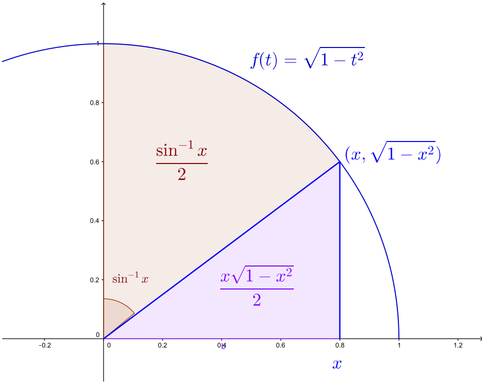


Figure: $\int_0^x \sqrt{1-t^2}dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2}$

Example

1. $$\int_1^3 (x^3 - 4x + 5)dx = \left[\frac{x^4}{4} - 2x^2 + 5x\right]_1^3 = \left[\left(\frac{3^4}{4} - 2(3^2) + 5(3)\right) - \left(\frac{1^4}{4} - 2(1^2) + 5(1)\right)\right] = 14$$

2. $$\int_{-3}^0 e^{2x+6}dx = \left[\frac{e^{2x+6}}{2}\right]_{-3}^0 = \frac{e^6 - 1}{2}$$

3. $$\int_0^{\frac{\pi}{12}} \sec^2 3x\,dx = \left[\frac{\tan 3x}{3}\right]_0^{\frac{\pi}{12}} = \frac{\tan 3(\frac{\pi}{12}) - \tan 0}{3} = \frac{1}{3}$$

The fundamental theorem of calculus can be used to evaluate limit of series of a certain form.

Theorem

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\ = & \lim_{n\to\infty} \frac{1}{n}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right) \\ = & \int_0^1 f(x)dx \end{aligned}$$

Example

Find

1. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)$$
2. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$$
3. $$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{n+k}}=\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{2n}}\right)$$

Example

1. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x}\,dx = [\ln(1+x)]_0^1 = \ln 2$$

2. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(\frac{k}{n})^2} = \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

3. $$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{n+k}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k}{n}}} = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx = [2\sqrt{1+x}]_0^1 = 2(\sqrt{2}-1)$$

Example

Find $\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}$.

Solution

$$\begin{aligned}
& \ln\left(\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}\right) \\
= \quad & \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{(n+1)(n+2)\cdots(2n)}{n^n}\right) \\
= \quad & \lim_{n\to\infty} \frac{1}{n} \ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right) \\
= \quad & \lim_{n\to\infty} \frac{1}{n} \left(\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\cdots+\ln\left(1+\frac{n}{n}\right)\right) \\
= \quad & \int_0^1 \ln(1+x)dx \\
= \quad & [(1+x)\ln(1+x)-x]_0^1 \\
= \quad & 2\ln 2 - 1
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n} = e^{2\ln 2-1} = \frac{4}{e} \approx 1.4715.$$

Example (Definite integral and substitution)

1. $\int_3^5 x\sqrt{x^2-9}\,dx$

Let $u = x^2 - 9$,

When $x = 3,\ u = 0$

When $x = 5,\ u = 16$

$du = 2xdx$

$$= \frac{1}{2}\int_0^{16} \sqrt{u}\,du$$

$$= \left[\frac{u^{\frac{3}{2}}}{3}\right]_0^{16}$$

$$= \frac{64}{3}$$

$$\int_3^5 x\sqrt{x^2-9}\,dx$$

$$= \frac{1}{2}\int_3^5 \sqrt{x^2-9}\,d(x^2-9)$$

$$= \frac{1}{3}\left[(x^2-9)^{\frac{3}{2}}\right]_3^5$$

$$= \frac{64}{3}$$

Example (Definite integral and substitution)

2. $$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$,

When $x = 0,\ u = 0$

When $x = \pi^2,\ u = \pi$

$$du = \frac{dx}{2\sqrt{x}}$$

$$= 2\int_0^{\pi} \sin u\,du$$

$$= 2\left[-\cos u\right]_0^{\pi}$$

$$= 4$$

$$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

$$= 2\int_0^{\pi^2} \sin\sqrt{x}\,d\sqrt{x}$$

$$= 2\left[-\cos\sqrt{x}\right]_0^{\pi^2}$$

$$= 2\left[-\cos\sqrt{\pi^2} - (-\cos 0)\right]$$

$$= 4$$

Example

We have the following formulas for derivatives of functions defined by integrals.

1. $$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$
2. $$\frac{d}{dx}\int_x^b f(t)dt = -f(x)$$
3. $$\frac{d}{dx}\int_a^{v(x)} f(t)dt = f(v)\frac{dv}{dx}$$
4. $$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

Proof.

1. This is the first part of fundamental theorem of calculus.
2. $$\frac{d}{dx}\int_x^b f(t)dt = \frac{d}{dx}\left(-\int_b^x f(t)dt\right) = -f(x)$$
3. $$\frac{d}{dx}\int_a^{v(x)} f(t)dt = \left(\frac{d}{dv}\int_a^{v(x)} f(t)dt\right)\frac{dv}{dx} = f(v)\frac{dv}{dx}$$
4. $$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = \frac{d}{dx}\left(\int_c^{v(x)} f(t)dt + \int_{u(x)}^{c} f(t)dt\right) = \frac{d}{dx}\left(\int_c^{v(x)} f(t)dt - \int_c^{u(x)} f(t)dt\right) = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

□

Example

Find $F'(x)$ for the the functions.

1. $F(x) = \int_1^x \sqrt{t}e^t dt$
2. $F(x) = \int_x^\pi \frac{\sin t}{t} dt$
3. $F(x) = \int_0^{\sin x} \sqrt{1+t^4} dt$
4. $F(x) = \int_{-x}^{x^2} e^{t^2} dt$

Solution

1. $$\frac{d}{dx}\int_1^x \sqrt{t}e^t dt = \sqrt{x}e^x$$

2. $$\frac{d}{dx}\int_x^\pi \frac{\sin t}{t} dt = -\frac{\sin x}{x}$$

3. $$\begin{aligned}\frac{d}{dx}\int_0^{\sin x} \sqrt{1+t^4} dt &= \sqrt{1+\sin^4 x}\frac{d}{dx}\sin x \\ &= \cos x\sqrt{1+\sin^4 x}\end{aligned}$$

4. $$\begin{aligned}\frac{d}{dx}\int_{-x}^{x^2} e^{t^2} dt &= e^{(x^2)^2}\frac{d}{dx}x^2 - e^{(-x)^2}\frac{d}{dx}(-x) \\ &= 2xe^{x^4} + e^{x^2}\end{aligned}$$

Trigonometric integrals

Techniques

Useful identities for trigonometric integrals.

1.
   - $\cos^2 x + \sin^2 x = 1$
   - $\sec^2 x = 1 + \tan^2 x$
   - $\csc^2 x = 1 + \cot^2 x$
2.
   - $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
   - $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
   - $\cos x \sin x = \dfrac{\sin 2x}{2}$
3.
   - $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$
   - $\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$
   - $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

Techniques

To evaluate

$$\int \cos^m x \sin^n x dx$$

where m, n *are non-negative integers,*

- *Case 1. If* m *is odd, use* $\cos x dx = d \sin x$*. (Substitute* $u = \sin x$*.)*
- *Case 2. If* n *is odd, use* $\sin x dx = -d \cos x$*. (Substitute* $u = \cos x$*.)*
- *Case 3. If both* m, n *are even, then use double angle formulas to reduce the power.*

$$\begin{aligned} \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos x \sin x &= \frac{\sin 2x}{2} \end{aligned}$$

Techniques

1. $$\int \tan x dx = \ln|\sec x| + C$$
2. $$\int \cot x dx = \ln|\sin x| + C$$
3. $$\int \sec x dx = \ln|\sec x + \tan x| + C$$
4. $$\int \csc x dx = \ln|\csc x - \cot x| + C$$

Proof

We prove (1), (3) and the rest are left as exercise.

$$1. \int \tan x dx = \int \frac{\sin x dx}{\cos x} = -\int \frac{d\cos x}{\cos x} = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$3. \int \sec x dx = \int \frac{\sec x(\sec x + \tan x)dx}{(\sec x + \tan x)} = \int \frac{(\sec^2 x + \sec x \tan x)dx}{(\sec x + \tan x)} = \int \frac{d(\tan x + \sec x)}{(\sec x + \tan x)} = \ln|\sec x + \tan x| + C$$

Techniques

To evaluate

$$\int \sec^m x \tan^n x dx$$

where m, n *are non-negative integers,*

- *Case 1. If* m *is even, use* $\sec^2 x dx = d \tan x$*. (Substitute* $u = \tan x$*.)*
- *Case 2. If* n *is odd, use* $\sec x \tan x dx = d \sec x$*. (Substitute* $u = \sec x$*.)*
- *Case 3. If both* m *is odd and* n *is even, use* $\tan^2 x = \sec^2 x - 1$ *to write everything in terms of* $\sec x$*.*

Example

Evaluate the following integrals.

1. $\int \sin^2 x dx$
2. $\int \cos^4 3x dx$
3. $\int \cos 2x \cos x dx$
4. $\int \cos 3x \sin 5x dx$

Solution

$$1.\ \int \sin^2 x dx = \int \left(\frac{1-\cos 2x}{2}\right) dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\begin{aligned} 2.\ \int \cos^4 x dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \left(\frac{1+2\cos 2x+\cos^2 2x}{4}\right) dx \\ &= \frac{x}{4} + \frac{\sin 2x}{4} + \int \left(\frac{1+\cos 4x}{8}\right) dx \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \end{aligned}$$

$$3.\ \int \cos 2x \cos x dx = \frac{1}{2}\int (\cos 3x + \cos x)\, dx = \frac{\sin 3x}{6} + \frac{\sin x}{2} + C$$

$$4.\ \int \cos 3x \sin 5x dx = \frac{1}{2}\int (\sin 8x + \sin 2x)\, dx = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$$

Example

Evaluate the following integrals.

1. $\displaystyle\int \cos x \sin^4 x dx$
2. $\displaystyle\int \cos^2 x \sin^3 x dx$
3. $\displaystyle\int \cos^4 x \sin^2 x dx$

Solution

$$1.\ \int \cos x \sin^4 x dx = \int \sin^4 x d \sin x = \frac{\sin^5 x}{5} + C$$

$$2.\ \int \cos^2 x \sin^3 x dx = -\int \cos^2 x(1 - cos^2 x) d \cos x$$

$$= -\int (\cos^2 x - cos^4 x) d \cos x$$

$$= -\frac{cos^3 x}{3} + \frac{cos^5 x}{5} C$$

$$3.\ \int \cos^4 x \sin^2 x dx = \int \left(\frac{1 + \cos 2x}{2}\right)\left(\frac{\sin 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int \left(\sin^2 2x + \cos 2x \sin^2 2x\right) dx$$

$$= \frac{1}{8}\int \left(\frac{1 - \cos 4x}{2}\right) dx + \frac{1}{16}\int \sin^2 2x d \sin 2x$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$$

Example

Evaluate the following integrals.

1. $\displaystyle\int \sec^2 x \tan^2 x dx$
2. $\displaystyle\int \sec x \tan^3 x dx$
3. $\displaystyle\int \tan^3 x dx$

Solution

$$1.\ \int \sec^2 x \tan^2 x dx = \int \tan^2 x d \tan x = \frac{\tan^3 x}{3} + C$$

$$2.\ \int \sec x \tan^3 x dx = \int \tan^2 x d \sec x = \int (\sec^2 x - 1) d \sec x$$

$$= \frac{\sec^3 x}{3} - \sec x + C$$

$$3.\ \int \tan^3 x dx = \int \tan x (\sec^2 x - 1) dx$$

$$= \int \tan x \sec^2 x dx - \int \tan x dx$$

$$= \int \tan x d \tan x - \ln|\sec x|$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

Integration by parts

Techniques

Suppose the integrand is of the form $u(x)v'(x)$. *Then we may evaluate the integration using the formula*

$$\int uv'dx = uv - \int u'vdx.$$

The above formula is called integration by parts. It is usually written in the form

$$\int udv = uv - \int vdu.$$

Example

Evaluate the following integrals.

1. $\displaystyle\int xe^{3x}dx$
2. $\displaystyle\int x^2 \cos x dx$
3. $\displaystyle\int x^3 \ln x dx$
4. $\displaystyle\int \ln x dx$

Solution

1. $$\int xe^{3x}dx = \frac{1}{3}\int xde^{3x} = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$$

2. $$\begin{aligned}\int x^2 \cos x dx &= \int x^2 d\sin x \\ &= x^2 \sin x - \int \sin x dx^2 \\ &= x^2 \sin x - 2\int x \sin x dx \\ &= x^2 \sin x + 2\int x d\cos x \\ &= x^2 \sin x + 2x\cos x - 2\int \cos x dx \\ &= x^2 \sin x + 2x \cos x - 2\sin x + C\end{aligned}$$

Solution

3. $$\int x^3 \ln x dx = \frac{1}{4}\int \ln x dx^4$$
$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^4 d\ln x$$
$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^4 \left(\frac{1}{x}\right) dx$$
$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 dx$$
$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

4. $$\int \ln x dx = x\ln x - \int x d\ln x$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + C$$

Example

Evaluate the following integrals.

5. $\int_0^{\pi} x \sin x \, dx$
6. $\int_0^1 e^{\sqrt{x}} dx$

Solution

$$
\begin{aligned}
5.\ \int_0^\pi x\sin x\,dx &= -\int_0^\pi x\,d\cos x \\
&= -[x\cos x]_0^\pi + \int_0^\pi \cos x\,dx \\
&= -(\pi\cos\pi - 0) + [\sin x]_0^\pi \\
&= \pi
\end{aligned}
$$

$$
\begin{aligned}
6.\ \int_0^1 e^{\sqrt{x}}dx &= 2\int_0^1 \sqrt{x}e^{\sqrt{x}}d\sqrt{x} \\
&= 2\int_0^1 \sqrt{x}de^{\sqrt{x}} \\
&= 2[\sqrt{x}e^{\sqrt{x}}]_0^1 - 2\int_0^1 e^{\sqrt{x}}d\sqrt{x} \\
&= 2e - 2[e^{\sqrt{x}}]_0^1 \\
&= 2e - 2(e-1) \\
&= 2
\end{aligned}
$$

Example

Evaluate the following integrals.

7. $\displaystyle\int \sin^{-1} x dx$
8. $\displaystyle\int \ln(1+x^2) dx$
9. $\displaystyle\int \sec^3 x dx$
10. $\displaystyle\int e^x \sin x dx$

Solution

$$7.\ \int \sin^{-1} x dx = x\sin^{-1} x - \int x d\sin^{-1} x$$
$$= x\sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}}$$
$$= x\sin^{-1} x + \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}}$$
$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

$$8.\ \int \ln(1+x^2) dx = x\ln(1+x^2) - \int x d\ln(1+x^2)$$
$$= x\ln(1+x^2) - 2\int \frac{x^2 dx}{1+x^2}$$
$$= x\ln(1+x^2) - 2\int \left(1 - \frac{1}{1+x^2}\right) dx$$
$$= x\ln(1+x^2) - 2x + 2\tan^{-1} x + C$$

Solution

9.
$$\begin{aligned}
\int \sec^3 x\,dx &= \int \sec x\,d\tan x \\
&= \sec x \tan x - \int \tan x\,d\sec x \\
&= \sec x \tan x - \int \sec x \tan^2 x\,dx \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx \\
\int \sec^3 x\,dx &= \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + C
\end{aligned}$$

Solution

10.
$$\begin{aligned}
\int e^x \sin x dx &= \int \sin x de^x \\
&= e^x \sin x - \int e^x d \sin x \\
&= e^x \sin x - \int e^x \cos x dx \\
&= e^x \sin x - \int \cos x de^x \\
&= e^x \sin x - e^x \cos x + \int e^x d \cos x \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x dx \\
2 \int e^x \sin x dx &= e^x \sin x - e^x \cos x + C' \\
\int e^x \sin x dx &= \frac{1}{2} (e^x \sin x - e^x \cos x) + C
\end{aligned}$$

Reduction formula

Techniques

For integral of the forms

$$\begin{aligned} I_n &= \int \cos^n x dx, \int \sin^n x dx, \int x^n \cos x dx, \int x^n \sin x dx, \\ & \int \sec^n x dx, \int \csc^n x dx, \int x^n e^x dx, \int (\ln x)^n dx, \\ & \int e^x \cos^n x dx, \int e^x \sin^n x dx, \int \frac{dx}{(x^2+a^2)^n}, \int \frac{dx}{(a^2-x^2)^n}, \end{aligned}$$

we may use integration by parts to find a formula to express I_n *in terms of* I_k *with* $k < n$. *Such a formula is called reduction formula.*

Example

Let

$$I_n = \int x^n \cos x dx$$

for positive integer n. Prove that

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}, \text{ for } n \geq 2$$

Proof.

$$\begin{aligned}
I_n = \int x^n \cos x dx &= \int x^n d \sin x \\
&= x^n \sin x - \int \sin x dx^n \\
&= x^n \sin x - n \int x^{n-1} \sin x dx \\
&= x^n \sin x + n \int x^{n-1} d \cos x \\
&= x^n \sin x + n x^{n-1} \cos x - n \int \cos x dx^{n-1} \\
&= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx \\
&= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}
\end{aligned}$$

□

Example

Let

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

where $a > 0$ is a positive real number for positive integer n. Prove that

$$I_n = \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, \text{ for } n \geq 2$$

Proof

$$
\begin{aligned}
I_n = \int \frac{dx}{(x^2+a^2)^n} &= \frac{x}{(x^2+a^2)^n} - \int x d\left(\frac{1}{(x^2+a^2)^n}\right) \\
&= \frac{x}{(x^2+a^2)^n} + \int \frac{2nx^2 dx}{(x^2+a^2)^{n+1}} \\
&= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{(x^2+a^2-a^2)dx}{(x^2+a^2)^{n+1}} \\
&= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{dx}{(x^2+a^2)^n} - 2na^2 \int \frac{dx}{(x^2+a^2)^{n+1}} \\
&= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1} \\
I_{n+1} &= \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n
\end{aligned}
$$

Replacing n *by* $n-1$, *we have*

$$I_n = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}.$$

Alternative proof.

$$
\begin{aligned}
I_n &= \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^n}dx \\
&= \frac{1}{a^2}\int \left(\frac{1}{(x^2+a^2)^{n-1}} - \frac{x^2}{(x^2+a^2)^n}\right) dx \\
&= \frac{1}{a^2}I_{n-1} - \frac{1}{2a^2}\int \frac{x}{(x^2+a^2)^n}d(x^2+a^2) \\
&= \frac{1}{a^2}I_{n-1} + \frac{1}{2(n-1)a^2}\int xd\left(\frac{1}{(x^2+a^2)^{n-1}}\right) \\
&= \frac{1}{a^2}I_{n-1} + \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} - \frac{1}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}} \\
&= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \left(\frac{1}{a^2} - \frac{1}{2(n-1)a^2}\right) I_{n-1} \\
&= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}I_{n-1}
\end{aligned}
$$

□

Example

Prove the following reduction formula

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

for $n \geq 2$. Hence show that

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \dfrac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3} & \text{when } n \text{ is odd} \\ \dfrac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\cdot\dfrac{\pi}{2} & \text{when } n \text{ is even} \end{cases}$$

Proof

$$
\begin{aligned}
\int \sin^n x dx &= -\int \sin^{n-1} x d\cos x \\
&= -\cos x \sin^{n-1} x + \int \cos x d \sin^{n-1} x \\
&= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x dx \\
n\int \sin^n x dx &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx \\
\int \sin^n x dx &= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx
\end{aligned}
$$

Proof

Hence when n is odd

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n x dx &= -\left[\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\
&= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\
&= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx \\
&\vdots \\
&= \frac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3}\int_0^{\frac{\pi}{2}} \sin x dx \\
&= \frac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3}
\end{aligned}$$

Proof.

when n is even

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n x dx &= -\left[\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\
&= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\
&= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx \\
&\vdots \\
&= \frac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\int_0^{\frac{\pi}{2}} dx \\
&= \frac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}
\end{aligned}$$

□

Example

$$I_n = \int x^n e^x dx; \qquad I_n = x^n e^x - nI_{n-1},\ n \geq 1$$

$$I_n = \int (\ln x)^n dx; \qquad I_n = x(\ln x)^n - nI_{n-1},\ n \geq 1$$

$$I_n = \int x^n \sin x dx; \qquad I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2},\ n \geq 2$$

$$I_n = \int \cos^n x dx; \qquad I_n = \frac{\cos^{n-1} x \sin x}{n} + (n-1)I_{n-2},\ n \geq 2$$

$$I_n = \int \sec^n x dx; \qquad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2},\ n \geq 2$$

$$I_n = \int e^x \cos^n x dx; \qquad I_n = \frac{e^x \cos^{n-1} x(\cos x + n \sin x)}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2},\ n \geq 2$$

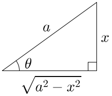
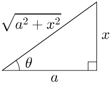
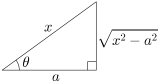
$$I_n = \int e^x \sin^n x dx; \qquad I_n = \frac{e^x \sin^{n-1} x(\sin x - n \cos x)}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2},\ n \geq 2$$

$$I_n = \int x^n \sqrt{x+a} dx; \qquad I_n = \frac{2x^n (x+a)^{\frac{3}{2}}}{2n+3} - \frac{2na}{2n+3} I_{n-1},\ n \geq 1$$

$$I_n = \int \frac{x^n}{\sqrt{x+a}} dx; \qquad I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2na}{2n+1} I_{n-1},\ n \geq 1$$

Trigonometric substitution

Techniques (Trigonometric substitution)

<table>
<tr><th>Expression</th><th>Substitution</th><th>dx</th><th>Trigonometric ratios</th><th></th></tr>
<tr><td>$\sqrt{a^2 - x^2}$</td><td>$x = a\sin\theta$</td><td>$dx = a\cos\theta d\theta$</td><td></td><td>$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$
$\sin\theta = \frac{x}{a}$
$\tan\theta = \frac{x}{\sqrt{a^2 - x^2}}$</td></tr>
<tr><td>$\sqrt{a^2 + x^2}$</td><td>$x = a\tan\theta$</td><td>$dx = a\sec^2\theta d\theta$</td><td></td><td>$\cos\theta = \frac{a}{\sqrt{a^2 + x^2}}$
$\sin\theta = \frac{x}{\sqrt{a^2 + x^2}}$
$\tan\theta = \frac{x}{a}$</td></tr>
<tr><td>$\sqrt{x^2 - a^2}$</td><td>$x = a\sec\theta$</td><td>$dx = a\sec\theta\tan\theta d\theta$</td><td></td><td>$\cos\theta = \frac{a}{x}$
$\sin\theta = \frac{\sqrt{x^2 - a^2}}{x}$
$\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$</td></tr>
</table>

Theorem

1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$
2. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
3. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \cos^{-1}\frac{a}{x} + C$$

Proof

1. *Let* $x = a\sin\theta$*. Then*

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta \\ dx &= a\cos\theta d\theta\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}}\,dx &= \int \frac{1}{a\cos\theta}(a\cos\theta d\theta) \\ &= \int d\theta \\ &= \theta + C \\ &= \sin^{-1}\frac{x}{a} + C\end{aligned}$$

Proof

2. *Let* $x = a\tan\theta$. *Then*

$$\begin{aligned} a^2 + x^2 &= a^2 + a^2\tan^2\theta = a^2\sec^2\theta \\ dx &= a\sec^2\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{a^2+x^2}\,dx &= \int \frac{1}{a^2\sec^2\theta}(a\sec^2\theta d\theta) \\ &= \frac{1}{a}\int d\theta \\ &= \frac{\theta}{a} + C \\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C \end{aligned}$$

Proof.

3. Let $x = a\sec\theta$. Then

$$\begin{aligned} x\sqrt{x^2-a^2} &= a\sec\theta\sqrt{a^2\sec^2\theta - a^2} = a^2\sec\theta\tan\theta \\ dx &= a\sec\theta\tan\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2-a^2}}\,dx &= \int \frac{1}{a^2\sec\theta\tan\theta}(a\sec\theta\tan\theta d\theta) \\ &= \frac{1}{a}\int d\theta \\ &= \frac{\theta}{a} + C \\ &= \frac{1}{a}\cos^{-1}\frac{a}{x} + C \end{aligned}$$

Note that $\theta = \cos^{-1}\frac{a}{x}$ since $\cos\theta = \frac{1}{\sec\theta} = \frac{a}{x}$.

□

Example

Use trigonometric substitution to evaluate the following integrals.

1. $\displaystyle\int \sqrt{1-x^2}\,dx$
2. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$
3. $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$
4. $\displaystyle\int \frac{1}{(9+x^2)^2}\,dx$

Solution

1. *Let* $x = \sin\theta$. *Then*

$$\begin{aligned} \sqrt{1-x^2} &= \sqrt{1-\sin^2\theta} = \cos\theta \\ dx &= \cos\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \sqrt{1-x^2}\, dx &= \int \cos^2\theta d\theta \\ &= \int \frac{\cos 2\theta + 1}{2} d\theta \\ &= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C \\ &= \frac{\sin\theta\cos\theta}{2} + \frac{\sin^{-1}x}{2} + C \\ &= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + C \end{aligned}$$

Solution

2. *Let* $x = \tan\theta$. *Then*

$$\begin{aligned} 1 + x^2 &= 1 + \tan^2\theta = \sec^2\theta \\ dx &= \sec^2\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sec x}(\sec^2\theta d\theta) \\ &= \int \sec\theta d\theta \\ &= \ln|\tan\theta + \sec\theta| + C \\ &= \ln(x + \sqrt{1+x^2}) + C \end{aligned}$$

Solution

3. *Let* $x = 2\sin\theta$. *Then*

$$\begin{aligned} \sqrt{4-x^2} &= \sqrt{4-4\sin^2\theta} = 2\cos\theta \\ dx &= 2\cos\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{8\sin^3\theta}{2\cos\theta}(2\cos\theta d\theta) \\ &= 8\int \sin^3\theta d\theta \\ &= -8\int (1-\cos^2\theta) d\cos\theta \\ &= 8\left(\frac{\cos^3\theta}{3} - \cos\theta\right) + C \\ &= \frac{(4-x^2)^{\frac{3}{2}}}{3} - 4(4-x^2)^{\frac{1}{2}} + C \end{aligned}$$

Solution

4. *Let* $x = 3\tan\theta$. *Then*

$$\begin{aligned} 9 + x^2 &= 9 + 9\tan^2\theta = 9\sec^2\theta \\ dx &= 3\sec^2\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{(9+x^2)^2}\,dx &= \int \frac{1}{81\sec^4\theta}(3\sec^2\theta d\theta) = \frac{1}{27}\int \cos^2\theta d\theta \\ &= \frac{1}{54}\int (\cos 2\theta + 1)d\theta = \frac{1}{54}\left(\frac{\sin 2\theta}{2} + \theta\right) + C \\ &= \frac{1}{54}(\cos\theta\sin\theta + \theta) + C \\ &= \frac{1}{54}\left(\frac{3}{\sqrt{9+x^2}} \cdot \frac{x}{\sqrt{9+x^2}} + \tan^{-1}\frac{x}{3}\right) + C \\ &= \frac{x}{18(9+x^2)} + \frac{1}{54}\tan^{-1}\frac{x}{3} + C \end{aligned}$$

Integration of rational functions

Definition (Rational functions)

A rational function is a function of the form

$$R(x) = \frac{f(x)}{g(x)}$$

where $f(x), g(x)$ are polynomials with real coefficients with $g(x) \neq 0$.

Techniques

We can integrate a rational function $R(x)$ with the following two steps.

1. *Find the partial fraction decomposition of $R(x)$, that is, express*

$$R(x) = q(x)+\sum \frac{A}{(x-\alpha)^k}+\sum \frac{B(x+a)}{((x+a)^2+b^2)^k}+\sum \frac{C}{((x+a)^2+b^2)^k}$$

where $q(x)$ is a polynomial, A, B, C, α, a, b represent real numbers and k represents positive integer.

2. *Integrate the partial fraction.*

Theorem

Let $R(x) = \dfrac{f(x)}{g(x)}$ *be a rational function. We may assume that the leading coefficient of* $g(x)$ *is 1.*

1. *(Division algorithm for polynomials) There exists polynomials* $q(x), r(x)$ *with* $\deg(r(x)) < \deg(g(x))$ *or* $r(x) = 0$ *such that*

   $$R(x) = q(x) + \frac{r(x)}{g(x)}.$$

   $q(x)$ *and* $r(x)$ *are the quotient and remainder of the division* $f(x)$ *by* $g(x)$*.*

2. *(Fundamental theorem of algebra for real polynomials)* $g(x)$ *can be written as a product of linear or quadratic polynomials. More precisely, there exists real numbers* $\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n, b_1, \ldots, b_n$ *and positive integers* $k_1, \ldots, k_m, l_1, \ldots, l_n$ *such that*

   $$g(x) = (x-\alpha_1)^{k_1} \cdots (x-\alpha_k)^{k_m}((x+a_1)^2+b_1^2)^{l_1} \cdots ((x+a_n)^2+b)_n^2)^{l_n}.$$

Techniques

Partial fractions can be integrated using the formulas below.

- $$\int \frac{dx}{(x-\alpha)^k} = \begin{cases} \ln|x-\alpha| + C, & \textit{if } k = 1 \\ -\dfrac{1}{(k-1)(x-\alpha)^{k-1}} + C, & \textit{if } k > 1 \end{cases}$$

- $$\int \frac{x\,dx}{(x^2+a^2)^k} = \begin{cases} \dfrac{1}{2}\ln(x^2+a^2) + C, & \textit{if } k = 1 \\ -\dfrac{1}{2(k-1)(x^2+a^2)^{k-1}} + C, & \textit{if } k > 1 \end{cases}$$

- $$\int \frac{dx}{(x^2+a^2)^k}$$

$$= \begin{cases} \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C, & \textit{if } k = 1 \\ \dfrac{x}{2a^2(k-1)(x^2+a^2)^{k-1}} + \dfrac{2k-3}{2a^2(k-1)}\displaystyle\int \frac{dx}{(x^2+a^2)^{k-1}}, & \textit{if } k > 1 \end{cases}$$

Theorem

Suppose $\dfrac{f(x)}{g(x)}$ *is a rational function such that the degree of* $f(x)$ *is smaller than the degree of* $g(x)$ *and* $g(x)$ *has only simple real roots, i.e.,*

$$g(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$$

for distinct real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$ *and* $a \neq 0$. *Then*

$$\frac{f(x)}{g(x)} = \frac{f(\alpha_1)}{g'(\alpha_1)(x-\alpha_1)} + \frac{f(\alpha_2)}{g'(\alpha_2)(x-\alpha_2)} + \cdots + \frac{f(\alpha_k)}{g'(\alpha_k)(x-\alpha_k)}$$

Proof

First, observe that

$$g'(x) = \sum_{j=1}^{k} a(x-\alpha_1)(x-\alpha_2)\cdots(\widehat{x-\alpha_j})\cdots(x-\alpha_k)$$

where $(\widehat{x-\alpha_i})$ *means the factor* $x-\alpha_i$ *is omitted. Thus we have*

$$\begin{aligned} g'(\alpha_i) &= \sum_{j=1}^{k} a(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots(\widehat{\alpha_i-\alpha_j})\cdots(\alpha_i-\alpha_k) \\ &= a(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots(\widehat{\alpha_i-\alpha_i})\cdots(\alpha_i-\alpha_k) \end{aligned}$$

Since $g(x)$ *has distinct real zeros, the partial fraction decomposition takes the form*

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_k}{x-\alpha_k}.$$

Proof.

Multiplying both sides by $g(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$, we get

$$f(x) = \sum_{i=1}^{k} A_i a(x-\alpha_1)(x-\alpha_2)\cdots\widehat{(x-\alpha_i)}\cdots(x-\alpha_k)$$

For $i = 1, 2, \cdots, k$, substituting $x = \alpha_i$, we obtain

$$\begin{aligned} f(\alpha_i) &= \sum_{j=1}^{k} A_j a(\alpha_j-\alpha_1)(\alpha_j-\alpha_2)\cdots\widehat{(\alpha_j-\alpha_i)}\cdots(\alpha_j-\alpha_k) \\ &= A_i a(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_i)}\cdots(\alpha_i-\alpha_k) \\ &= A_i g'(\alpha_i) \end{aligned}$$

and the result follows. □

Example

Evaluate the following integrals.

1. $$\int \frac{x^5 + 2x - 1}{x^3 - x}\,dx$$
2. $$\int \frac{9x - 2}{2x^3 + 3x^2 - 2x}\,dx$$
3. $$\int \frac{x^2 - 2}{x(x-1)^2}\,dx$$
4. $$\int \frac{x^2}{x^4 - 1}\,dx$$
5. $$\int \frac{8x^2}{x^4 + 4}\,dx$$
6. $$\int \frac{2x + 1}{x^4 + 2x^2 + 1}\,dx$$

Solution

1. *By division and factorization* $x^3 - x = x(x-1)(x+1)$, *we obtain the partial fraction decomposition*

$$\frac{x^5 + 4x - 3}{x^3 - x} = x^2 + 1 + \frac{5x - 3}{x^3 - x} = x^2 + 1 + \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Multiply both sides by $x(x-1)(x+1)$ *and obtain*

$$\begin{aligned} & 5x - 3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1) \\ \Rightarrow \quad & A = 3, B = 1, C = -4. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^5 + 4x - 3}{x^3 - x}\,dx &= \int \left(x^2 + 1 + \frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1} \right) dx \\ &= \frac{x^3}{3} + x + 3\ln|x| + \ln|x-1| - 4\ln|x+1| + C. \end{aligned}$$

Solution

2. *By factorization* $2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$, *we obtain the partial fraction decomposition*

$$\frac{9x-2}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}.$$

Multiply both sides by $x(x+2)(2x-1)$ *and obtain*

$$\begin{aligned} & 9x - 2 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2) \\ \Rightarrow \quad & A = 1, B = -2, C = 2. \end{aligned}$$

Therefore

$$\begin{aligned} & \int \frac{9x-2}{2x^3+3x^2-2x}\,dx \\ = \quad & \int \left(\frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1} \right) dx \\ = \quad & \ln|x| - 2\ln|x+2| + \ln|2x-1| + C. \end{aligned}$$

Solution

3. *The partial fraction decomposition is*

$$\frac{x^2-2}{x(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x}.$$

Multiply both sides by $x(x-1)^2$ *and obtain*

$$\begin{aligned} & x^2-2 = Ax + Bx(x-1) + C(x-1)^2 \\ \Rightarrow \quad & A=-1, B=3, C=-2. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^2-2}{x(x-1)^2}\,dx &= \int \left(-\frac{1}{(x-1)^2} + \frac{3}{x-1} - \frac{2}{x}\right) dx \\ &= \frac{1}{x-1} + 3\ln|x-1| - 2\ln|x| + C. \end{aligned}$$

Solution

4. *The partial fraction decomposition is*

$$\begin{aligned}\frac{x^2}{x^4-1} &= \frac{x^2}{(x^2-1)(x^2+1)} \\ &= \frac{1}{2}\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right) \\ &= \frac{1}{2(x-1)(x+1)}+\frac{1}{2(x^2+1)} \\ &= \frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{2(x^2+1)}\end{aligned}$$

Therefore

$$\begin{aligned}\int\frac{x^2dx}{x^4-1} &= \int\left(\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{2(x^2+1)}\right)dx \\ &= \frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|+\frac{1}{2}\tan^{-1}x+C\end{aligned}$$

Solution

5. *By factorization* $x^4+4=(x^2+2)^2-(2x)^2=(x^2-2x+2)(x^2+2x+2)$,

$$\begin{aligned}
&\int \frac{8x^2}{x^4+4}\,dx \\
=&\int \frac{8x^2dx}{(x^2-2x+2)(x^2+2x+2)}\,dx \\
=&\int 2x\left(\frac{4x}{(x^2-2x+2)(x^2+2x+2)}\right)dx \\
=&\int 2x\left(\frac{1}{x^2-2x+2}-\frac{1}{x^2+2x+2}\right)dx \\
=&\int \left(\frac{2x}{(x-1)^2+1}-\frac{2x}{(x+1)^2+1}\right)dx \\
=&\int \left(\frac{2(x-1)}{(x-1)^2+1}+\frac{2}{(x-1)^2+1}-\frac{2(x+1)}{(x+1)^2+1}+\frac{2}{(x+1)^2+1}\right)dx \\
=&\ \ln(x^2-2x+2)+2\tan^{-1}(x-1)-\ln(x^2+2x+2)+2\tan^{-1}(x+1)+C
\end{aligned}$$

Solution

6.
$$
\begin{aligned}
& \int \frac{2x+1}{x^4+2x^2+1}\,dx \\
= & \int \frac{2x\,dx}{(x^2+1)^2} + \int \frac{dx}{(x^2+1)^2} \\
= & \int \frac{d(x^2+1)}{(x^2+1)^2} + \int \frac{x^2+1}{(x^2+1)^2}\,dx - \int \frac{x^2\,dx}{(x^2+1)^2} \\
= & -\frac{1}{x^2+1} + \int \frac{dx}{x^2+1} - \frac{1}{2}\int \frac{x\,d(x^2+1)}{(x^2+1)^2} \\
= & -\frac{1}{x^2+1} + \tan^{-1} x + \frac{1}{2}\int x\,d\left(\frac{1}{x^2+1}\right) \\
= & -\frac{1}{x^2+1} + \tan^{-1} x + \frac{1}{2}\left(\frac{x}{x^2+1}\right) - \frac{1}{2}\int \frac{dx}{x^2+1} \\
= & \frac{x-2}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C
\end{aligned}
$$

Example

Find the partial fraction decomposition of the following functions.

1. $$\frac{5x-3}{x^3-x}$$
2. $$\frac{9x-2}{2x^3+3x^2-2x}$$

Solution

1. *For* $g(x) = x^3 - x = x(x-1)(x+1)$, $g'(x) = 3x^2 - 1$. *Therefore*

$$\begin{aligned} \frac{5x-3}{x^3-x} &= \frac{-3}{g'(0)x} + \frac{5(1)-3}{g'(1)(x-1)} + \frac{5(-1)-3}{g'(-1)(x+1)} \\ &= \frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1} \end{aligned}$$

2. *For* $g(x) = 2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$, $g'(x) = 6x^2 + 6x - 2$. *Therefore*

$$\begin{aligned} &\frac{9x-2}{2x^3+3x^2-2x} \\ = \; &\frac{-2}{g'(0)x} + \frac{9(-2)-2}{g'(-2)(x+2)} + \frac{9(\frac{1}{2})-2}{g'(\frac{1}{2})(2x-1)} \\ = \; &\frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1} \end{aligned}$$

□

t-substitution

Techniques

To evaluate

$$\int R(\cos x, \sin x, \tan x)dx$$

where R is a rational function, we may use t-substitution

$$t = \tan \frac{x}{2}.$$

Then

$$\tan x = \frac{2t}{1-t^2};\ \cos x = \frac{1-t^2}{1+t^2};\ \sin x = \frac{2t}{1+t^2};$$
$$dx = d(2\tan^{-1} t) = \frac{2dt}{1+t^2}.$$

We have

$$\int R(\cos x, \sin x, \tan x)dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2dt}{1+t^2}$$

which is an integral of rational function.

Example

Use t-substitution to evaluate the following integrals.

1. $\displaystyle\int \frac{dx}{1+\cos x}$
2. $\displaystyle\int \frac{\sin x dx}{\cos x+\sin x}$
3. $\displaystyle\int \frac{dx}{1+\cos x+\sin x}$

Solution

1. *Let* $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$. *We have*

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \left(\frac{1}{1+\frac{1-t^2}{1+t^2}}\right)\frac{2dt}{1+t^2} = \int dt = t + C = \tan\frac{x}{2} + C \\ &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} + C = \frac{2\cos\frac{x}{2}\sin\frac{x}{2}}{2\cos^2\frac{x}{2}} + C = \frac{\sin x}{1+\cos x} + C\end{aligned}$$

Alternatively

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \frac{dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}\int \sec^2\frac{x}{2}dx \\ &= \tan\frac{x}{2} + C = \frac{\sin x}{1+\cos x} + C\end{aligned}$$

Solution

2. Let $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$. *We have*

$$\begin{aligned}
\int \frac{\sin x dx}{\cos x + \sin x} &= \int \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \frac{2dt}{1+t^2} \\
&= \int \left(\frac{1}{1+t^2} + \frac{t}{1+t^2} + \frac{t-1}{1+2t-t^2} \right) dt \\
&= \tan^{-1} t + \frac{1}{2} \ln|1+t^2| - \frac{1}{2} \ln|1+2t-t^2| + C \\
&= \tan^{-1} t - \frac{1}{2} \ln\left|\frac{1+2t-t^2}{1+t^2}\right| + C \\
&= \tan^{-1} t - \frac{1}{2} \ln\left|\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}\right| + C \\
&= \frac{x}{2} - \frac{1}{2} \ln|\cos x + \sin x| + C
\end{aligned}$$

Solution

Alternatively

$$\begin{aligned}\int \frac{\sin x dx}{\cos x + \sin x} &= \frac{1}{2}\int \left(1 - \frac{\cos x - \sin x}{\cos x + \sin x}\right) dx \\ &= \frac{x}{2} - \frac{1}{2}\int \frac{d(\sin x + \cos x)}{\cos x + \sin x} \\ &= \frac{x}{2} - \frac{1}{2}\ln|\cos x + \sin x| + C\end{aligned}$$

Solution

3. *Let* $t = \tan\dfrac{x}{2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$, $\sin x = \dfrac{2t}{1+t^2}$, $dx = \dfrac{2dt}{1+t^2}$. *We have*

$$\begin{aligned}
\int \frac{dx}{1+\cos x+\sin x} &= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \\
&= \int \frac{dt}{1+t} \\
&= \ln|1+t| + C \\
&= \ln\left|1+\tan\frac{x}{2}\right| + C \\
&= \ln\left|1+\frac{\sin x}{1+\cos x}\right| + C \\
&= \ln\left|\frac{1+\cos x+\sin x}{1+\cos x}\right| + C
\end{aligned}$$