Complex Fourier Series

<u>Def</u>: (1) A <u>complex trigonometric series</u> is a series of the form
$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

($\{c_n\}_{-\infty}^{\infty}$ is called a <u>bisequence</u> of cpx numbers & $\{c_n e^{inx}\}_{n=-\infty}^{\infty}$ is a <u>bisequence</u> of cpx-valued functions)

(2) $\sum_{-\infty}^{\infty} c_n e^{inx}$ is said to be <u>convergent at x</u> if $\lim_{N\to+\infty} \sum_{n=-N}^{N} c_n e^{inx}$ exists.

<u>Def</u>: <u>Complex Fourier Series</u> of a 2π-periodic cpx-valued function f which is <u>integrable</u> on $[-\pi,\pi]$, denoted by
$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$
is a cpx trigonometric series with (complex) Fourier coefficients c_n defined by
$$\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx, \quad \forall n\in\mathbb{Z}}$$

Notes (i) For cpx-valued function $f = u + iv$ with u, v real-valued

$$\int_a^b f \overset{\text{def}}{=} \left(\int_a^b u\right) + i\left(\int_a^b v\right)$$

(ii) f is called integrable $\Leftrightarrow$ both u & v are integrable.

Motivation for cpx Fourier Series:

"If" $\quad f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ & "converges nicely"

Then

$$f(x) e^{-imx} = \sum_{-\infty}^{\infty} c_n e^{inx} e^{-imx}$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) e^{-imx}\, dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

It is easy to find

$$\int_{-\pi}^{\pi} e^{i(n-m)x} dx = \begin{cases} 2\pi, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

Hence

$$\int_{-\pi}^{\pi} f(x) e^{-imx} dx = c_m \cdot 2\pi$$

$$\Rightarrow \qquad c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

Relationship between (Real) Fourier Series & complex Fourier Series for a real-valued function f.

By $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx - i\sin nx\right)dx$$
$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right) - \frac{1}{2}i\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)$$

Therefore:

for $n=0$, $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0$

for $n \geq 1$, $c_n = \frac{a_n}{2} - i\frac{b_n}{2}$

for $n \leq -1$, then $(-n) \geq 1$ &

$$c_n = \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(-n)x\,dx\right) + \frac{1}{2}i\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(-n)x\,dx\right)$$
$$= \frac{1}{2}a_{(-n)} + i\,\frac{1}{2}b_{(-n)}$$

$\therefore$ $$\boxed{c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{for } n \geq 1 \\ a_0 & \text{for } n = 0 \\ \frac{1}{2}(a_{(-n)} + i\,b_{(-n)}) & \text{for } n \leq -1 \end{cases}}$$ for real valued function.

Corollary: If f is a real-valued function, then

$$C_{-n} = \overline{C_n} \quad \leftarrow \text{cpx conjugate}$$

$\forall n \in \mathbb{Z}$.

(i.e. $C_n = \overline{C_{-n}}$)

Pf: $n \geq 1 \Rightarrow (-n) \leq -1$

$$\therefore \quad C_{-n} = \frac{1}{2}\left(a_{-(-n)} + i\, b_{-(-n)}\right)$$

$$= \frac{1}{2}(a_n + i b_n) = \overline{C_n}$$

Similar for others. $\cancel{\times}$

Prop: Let f be a 2π-periodic $\overbrace{\text{}}^{\text{real}}$ function which is differentiable on $[-\pi, \pi]$ with f' integrable on $[-\pi, \pi]$. Denote the Fourier coefficients of f & f' by $\{a_n(f), b_n(f);\ c_n(f)\}$ & $\{a_n(f'), b_n(f');\ c_n(f')\}$ respectively.

Then

$$\begin{cases} a_n(f') = n\, b_n(f) \\ b_n(f') = -n\, a_n(f) \end{cases}$$

&

$$c_n(f') = i n\, c_n(f)$$

(So it is more convenient to work with cpx Fourier coefficients when derivatives involved.)

Pf: $a_n(f') = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx$

(integration by-parts) $= \frac{1}{\pi}\left[f(x)\cos nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)(-n\sin nx)dx \right]$

$(f(\pi)=f(-\pi))$ $= \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = n\,b_n(f)$

Similarly for $b_n(f') = -n\,a_n(f)$ (Check)

For $c_n(f')$, either from the above formula relating c_n to a_n & b_n, or integration by parts directly

$$c_n(f') = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx = \frac{1}{2\pi}\left[\cancel{f(x)e^{-inx}}^{\,0}\Big|_{-\pi}^{\pi} + (in)\int_{-\pi}^{\pi} f(x)e^{-inx}dx \right]$$

$= in\,c_n(f)$. ※