THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Tutorial 7 (April 12)

The following were discussed in the tutorial this week.

Let E be a measurable subset of $\mathbb{R}$.

1. Let $1 \leq p < \infty$. Recall that

$$L^p(E) = \left\{ f \text{ measurable function on } E : \|f\|_p := \left(\int_E |f|^p \right)^{1/p} < \infty \right\},$$

   where we identify functions that are equal almost everywhere on E. Then $(L^p(E), \|\cdot\|_p)$ is a normed vector space.

2. Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
   (a) Prove the Young's inequality: $a \cdot b \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ for $a, b \geq 0$.
   (b) Prove the Hölder's inequality using the Young's inequality.
   (c) Deduce the Minkowski's inequality from the Hölder's inequality.

3. Let

$$L^\infty(E) = \{\text{bounded measurable functions on } E\},$$

   where we again identify functions that are equal almost everywhere on E. Define

$$\|f\|_\infty := \inf\{\alpha : m(\{x \in E : |f(x)| > \alpha\}) = 0\}.$$

   Then it is easy to see that

$$\|f\|_\infty \leq \lambda \quad \Longleftrightarrow \quad |f(x)| \leq \lambda \text{ for a.e. } x \in E.$$

   Moreover $(L^\infty(E), \|\cdot\|_\infty)$ is also a normed vector space.

4. Recall the Riesz-Fisher Theorem: For $1 \leq p \leq \infty$, $(L^p(E), \|\cdot\|_p)$ is complete.

5. Let $E = (0, \infty)$. Show that $L^1(E) \not\subseteq L^2(E)$ and $L^2(E) \not\subseteq L^1(E)$.

6. Suppose $m(E) < \infty$. Show that if $1 \leq r < s \leq \infty$, then $L^s(E) \subseteq L^r(E)$.

7. Suppose $\|f\|_r < \infty$ for some $1 \leq r < \infty$. Show that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$