THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Tutorial 6 (March 29)

The following were discussed in the tutorial this week.

Theorem 1 (Fundamental Theorem of Calculus Part I)**.** *Let* $f \in \mathcal{L}[a,b]$*. Then*

$$\frac{d}{dx}\int_a^x f = f(x) \quad \text{for a.e. } x \in [a,b].$$

Theorem 2 (Fundamental Theorem of Calculus Part II)**.** *If* $F \in \mathrm{ABC}[a,b]$*, then* $F'(x)$ *exists a.e. in* $[a,b]$*,* $F' \in \mathcal{L}[a,b]$ *and*

$$F(x) = \int_a^x F' + F(a) \quad \text{for all } x \in [a,b].$$

Conversely, if $f \in \mathcal{L}[a,b]$*, then the "indefinite integral" defined by*

$$x \mapsto \int_a^x f + \text{constant}$$

is absolutely continuous on $[a,b]$*.*

Exercise 1. Let $f \in \mathrm{BV}[a,b]$. Show that

(a) $\displaystyle\int_a^b |f'| \leq T_a^b(f)$;

(b) furthermore, $f \in \mathrm{ABC}[a,b]$ if and only if $\displaystyle\int_a^b |f'| = T_a^b(f)$.

Theorem 3 (Banach-Zarecki)**.** *Let* $f : [a,b] \to \mathbb{R}$*. Then* $f \in \mathrm{ABC}[a,b]$ *if and only if the following conditions are all satisfied.*

1. f *is continuous on* $[a,b]$*.*
2. $f \in \mathrm{BV}[a,b]$*.*
3. f *has Lusin N property, i.e.* f *maps sets of measure zero to sets of measure zero.*

Exercise 2. If $f \in \mathrm{ABC}[a,b]$, show that f maps measurable sets to measurable sets.

Exercise 3. Suppose f be absolutely continuous on $[\varepsilon, 1]$ for all $\varepsilon > 0$.

(a) If f is continuous at 0, is f absolutely continuous on $[0,1]$?

(b) Suppose further that f is of bounded variation on $[0,1]$, show that f is absolutely continuous on $[0,1]$.